

Algebraic methods in the study of systems of the reaction-diffusion type

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Abstract

Nonlinear systems of the reaction-diffusion type, including Gierer-Meinhardt models of autocatalysis, are studied by using Lie algebras coming from the prolongation structure. The consequences of this analytical approach, as the determination of special exact solutions, are compared with the corresponding results obtained via numerical simulations.

I. INTRODUCTION

Models for biological pattern formation may be described by reaction-diffusion (RD) equations of the type [1]

$$u_{kt} = \alpha_k \nabla^2 u_k + R_k(u), \quad (1.1)$$

(no sum over k), where $u = \{u_k\}$ are the dynamical fields, $k = 1, 2, \dots, N$, α_k are constants and $R_k(u)$ are functions of the fields which characterize the reactions among them.

The class of Eq. (1.1) includes popular models of chemical reactions [3]. In particular, when the diffusion coefficients α_k are not all equal, Turing instabilities may occur and complicated patterns emerge which have been related to morphogenesis, reaction front dynamics, self-organization and so forth [4].

Models of pattern generation in complex organisms are investigated mostly via computer simulations. A minor attention is paid to the analysis of the mathematical properties of the underlying equations. The knowledge of these properties, combined with numerical studies, could be of help both to compare theory and experiment, to check the validity of the models, and to suggest possible improvements.

Keeping in mind this programme, in this paper we apply the prolongation technique [5] to the $1 + 1$ dimensional RD system

$$u_{kt} = \alpha_k u_{kxx} + R_k(u), \quad (1.2)$$

($k = 1, 2, \dots, N$), where first the reaction functions $R_k(u)$ are not specified. Our approach contemplates the use of the concept of *pseudopotential*, that is an M -component vector (M arbitrary) $y = \{y(x, t)\} = (y_1, \dots, y_M)$ defined by

$$y_x = f^j(u) T_j y, \quad (1.3a)$$

$$y_t = g^j(u, u_x) T_j y, \quad (1.3b)$$

where summation over repeated indices is understood, $u = \{u_k\}$, $u_x = \{u_{kx}\}$, T_j are $M \times M$ matrices spanning a Lie algebra \mathcal{L}

$$[T_i, T_j] = c_{ij}^k T_k, \quad (1.4)$$

and f^j , g^j are functions to be determined in such a way that the compatibility condition $y_{tx} = y_{xt}$ reproduces Eq. (1.2). When this feature occurs, then Eqs. (1.3) represent a linearization of Eq. (1.2).

The motivation for dealing with Eq. (1.2) instead of Eq. (1.1) within the prolongation scheme is at least threefold. First, the prolongation machinery in $2 + 1$ dimensions is not yet well established, even if some examples exist concerning a few integrable cases [6]. Second, biological structures modeled by equations of the class (1.2) are interesting also in $1 + 1$ dimensions. One of them, which consists of Eqs. (1.2) where $R_k(u)$ is assumed to be linear in the fields $\{u_k\}$ ($k = 1, 2$), has been discussed recently by Kondo and Asai in relation to the stripe pattern mechanism of the angelfish *Pomachantus* [7]. Third, the boundary of a complex two-dimensional structure hosts a restricted $1 + 1$ dimensional RD system [8].

The main results achieved in this work are listed in correspondence of the contents of the different Sections in which the paper is planned.

In Sec. II, we deal with the prolongation of RD systems described by Eq. (1.2). In Sec. III, we study a class of quadratic models which are completely linearizable and provide several exact analytical solutions of the travelling wave type. These solutions are obtained by direct inspection. In Sec. IV, we emphasize the role of linearizability. The evolution equations for the pseudopotential are written explicitly and solved showing how new solutions can emerge in terms of known ones. In Sec. V, we apply the general results found in Sec. II to the prolongation of the Gierer-Meinhardt models [1]. The resulting algebra turns out to be closed and is that of the similitude group in the plane [2]. In Sec. VI, we explore the consequences of this underlying algebraic structure. In particular, we treat analytical approximations of particular classes of solutions, namely homogeneous and of the travelling wave type. Section VII contains results obtained via numerical integration of the evolution equations in order to check the analytical predictions and to suggest possible developments of the analysis. Finally, in Sec. VIII some concluding remarks are considered, while in the Appendices details of the calculations are reported.

II. PROLONGATION OF GENERAL RD SYSTEMS

Following the general strategy outlined in the Introduction, we begin our analysis of the constraints on the algebra \mathcal{L} by writing explicitly the compatibility condition $y_{tx} = y_{xt}$ of Eqs. (1.3), namely

$$\sum_k \left\{ \frac{\partial f^\gamma}{\partial u_k} u_{kt} - \frac{\partial g^\gamma}{\partial u_k} u_{kx} - \frac{\partial g^\gamma}{\partial u_{kx}} u_{kxx} \right\} + c_{\alpha\beta}^\gamma f^\alpha g^\beta = 0. \quad (2.1)$$

Substitution from Eq. (1.2) into Eq. (2.1) yields

$$\frac{\partial f^\gamma}{\partial u_k} (\alpha_k u_{kxx} + R_k) - \frac{\partial g^\gamma}{\partial u_k} u_{kx} - \frac{\partial g^\gamma}{\partial u_{kx}} u_{kxx} + c_{\alpha\beta}^\gamma f^\alpha g^\beta = 0. \quad (2.2)$$

The coefficient of u_{kxx} must vanish

$$\alpha_k \frac{\partial f^\gamma}{\partial u_k} = \frac{\partial g^\gamma}{\partial u_{kx}}, \quad (2.3)$$

so that

$$g^\gamma = \alpha_k \frac{\partial f^\gamma}{\partial u_k} u_{kx} + \Delta^\gamma(u), \quad (2.4)$$

where Δ^γ is a function of integration. Putting this result back into Eq. (2.2), we find

$$\frac{\partial f^\gamma}{\partial u_k} R_k - \alpha_k \frac{\partial^2 f^\gamma}{\partial u_k \partial u_l} u_{kx} u_{lx} - \frac{\partial \Delta^\gamma}{\partial u_k} u_{kx} + c_{\alpha\beta}^\gamma f^\alpha \alpha_k \frac{\partial f^\beta}{\partial u_k} u_{kx} + c_{\alpha\beta}^\gamma f^\alpha \Delta^\beta = 0. \quad (2.5)$$

Consequently, the following three conditions

$$(\alpha_k + \alpha_l) \frac{\partial^2 f^\gamma}{\partial u_k \partial u_l} = 0, \quad (2.6a)$$

$$\frac{\partial f^\gamma}{\partial u_k} R_k + c_{\alpha\beta}^\gamma f^\alpha \Delta^\beta = 0, \quad (2.6b)$$

$$\frac{\partial \Delta^\gamma}{\partial u_k} - c_{\alpha\beta}^\gamma f^\alpha \alpha_k \frac{\partial f^\beta}{\partial u_k} = 0, \quad (2.6c)$$

hold. Equations (2.6) entail two different cases distinguished by the condition

$$\alpha_k + \alpha_l \neq 0 \quad \forall k, l, \quad (2.7)$$

or

$$\alpha_k + \alpha_l = 0 \quad \text{for some } k, l. \quad (2.8)$$

Let us call “non-degenerate” and “degenerate” the problems corresponding to Eq. (2.7) and Eq. (2.8), respectively. In these cases, we see that f is linear in the fields u and, therefore, the structure of f and g may be expressed by

$$f^\alpha = H_k^\alpha u_k + K^\alpha, \quad (2.9a)$$

$$g^\alpha = \alpha_k H_k^\alpha u_{kx} + \Delta^\alpha(u), \quad (2.9b)$$

where H_k^α and K^α are constants. Inserting these quantities into Eq. (2.6c) we obtain

$$\frac{\partial \Delta^\gamma}{\partial u_k} = c_{\alpha\beta}^\gamma (H_l^\alpha u_l + K^\alpha) \alpha_k H_k^\beta = \alpha_k c_{\alpha\beta}^\gamma H_l^\alpha H_k^\beta u_l + c_{\alpha\beta}^\gamma K^\alpha H_k^\beta \alpha_k. \quad (2.10)$$

On the other hand, the compatibility condition

$$\frac{\partial^2 \Delta^\gamma}{\partial u_k \partial u_l} = \frac{\partial^2 \Delta^\gamma}{\partial u_l \partial u_k}, \quad (2.11)$$

gives

$$c_{\alpha\beta}^\gamma H_l^\alpha H_k^\beta \alpha_k = c_{\alpha\beta}^\gamma H_k^\alpha H_l^\beta \alpha_l \Rightarrow (\alpha_k + \alpha_l) c_{\alpha\beta}^\gamma H_k^\alpha H_l^\beta = 0. \quad (2.12)$$

From this relation, we get

$$c_{\alpha\beta}^\gamma H_k^\alpha H_l^\beta = 0 \quad (2.13)$$

in the non-degenerate case. Now we can exploit this equation to determine the Δ^γ term, i.e.

$$\frac{\partial \Delta^\gamma}{\partial u_k} = c_{\alpha\beta}^\gamma K^\alpha H_k^\beta \alpha_k \Rightarrow \Delta^\gamma = c_{\alpha\beta}^\gamma K^\alpha H_k^\beta \alpha_k + D^\gamma, \quad (2.14)$$

(D^γ being constants of integration), and the final expressions for f and g , which turn out to be

$$f^\alpha = H_k^\alpha u_k + K^\alpha, \quad (2.15a)$$

$$g^\alpha = \alpha_k H_k^\alpha u_{kx} + c_{\beta\gamma}^\alpha H_k^\gamma \alpha_k u_k + D^\alpha. \quad (2.15b)$$

At this stage the reaction terms come into play. All the unknown constants must be chosen in order to satisfy the final equation

$$\frac{\partial f^\gamma}{\partial u_k} R_k + c_{\alpha\beta}^\gamma f^\alpha \Delta^\beta = 0. \quad (2.16)$$

To recognize the algebraic structure of the problem, let us adopt a notation in which the Lie algebra indices are understood. We define

$$f = f^\alpha T_\alpha, \quad g = g^\alpha T_\alpha, \quad (2.17a)$$

$$K = K^\alpha T_\alpha, \quad D = D^\alpha T_\alpha, \quad (2.17b)$$

$$\Phi = H_k u_k = H_k^\alpha u_k T_\alpha, \quad (2.17c)$$

$$\Psi = \alpha_k H_k u_k = \alpha_k H_k^\alpha u_k T_\alpha, \quad (2.17d)$$

and write the linearized problem in the compact form

$$f = \Phi + K, \quad (2.18a)$$

$$g = \Psi_x + [K, \Psi] + D, \quad (2.18b)$$

where

$$[H_k, H_l] = 0, \quad (2.19a)$$

$$\sum_k H_k R_k + [\Phi + K, [K, \Psi] + D] = 0. \quad (2.19b)$$

Equations (2.19) must be satisfied whenever the fields u are chosen. Taking the independent monomials in the fields, we obtain an incomplete Lie algebra, in the sense that not all the commutators are known. By construction, we know that, if the incomplete algebra is satisfied, then the compatibility condition for the linearized problem is assured as long as the evolution equations hold.

On the other hand, the explicit form of the compatibility condition $y_{xt} = y_{tx}$ is easily found to be

$$\Phi_t - \Psi_{xx} + [\Phi + K, [K, \Psi] + D] = 0, \quad (2.20)$$

which can be written as

$$\Phi_t = \Psi_{xx} + \sum_k H_k R_k \quad (2.21)$$

by virtue of Eq. (2.19b). If the elements $\{H_k\}$ of \mathcal{L} were independent, then we could project Eq. (2.21) onto its components and recover the full set of evolution equations establishing the complete linearization of the reaction-diffusion system under consideration, i.e.

$$y_{tx} = y_{xt} \leftrightarrow \text{evolution equations (1.2)}. \quad (2.22)$$

However, as we said before, $\{H_k\}$ are subject to the incomplete Lie algebra (2.19) which is a severe constraint. Actually, Eqs. (2.19) say that every nonlinear term higher than a cubic polynomial in Eq. (1.2), implies a linear dependence among the elements of the set $\{H_k\}$. In other words, we have

$$\text{evolution equations} \rightarrow y_{tx} = y_{xt}, \quad (2.23a)$$

$$y_{tx} = y_{xt} \rightarrow \text{linear combinations of evolution equations}, \quad (2.23b)$$

$$\text{at most quadratic in the fields.} \quad (2.23c)$$

Therefore, the full equivalence expressed by Eq. (2.22) may be achieved only if (i) the reaction terms are at most quadratic or (ii) in the special degenerate cases where f and g are no more constrained to be linear in the fields. In all the other cases, Eqs. (1.3) represent only a semi-linearization of the evolution equations, i.e. they are equivalent only to a reduced set of linear combinations of the original equations. For simplicity, also in these cases we shall keep calling Eqs. (1.3) the *linearized problem* for Eq. (1.2).

We are interested in applications of the reaction-diffusion systems in problems related to morphogenesis, chemical autocatalysis and biological modeling. In these contexts negative diffusion coefficients do not admit a simple interpretation and therefore we shall not consider in detail the degenerate case. To this regard, we limit ourselves to mention the system

$$u_{1t} - u_{1xx} + 2u_1^2 u_2 - 2au_1 = 0, \quad (2.24a)$$

$$u_{2t} + u_{2xx} - 2u_1 u_2^2 + 2au_2 = 0, \quad (2.24b)$$

where a is a fixed constant. Equations (2.24), which admit an infinite dimensional prolongation Lie algebra endowed with a loop structure [9], are integrable, and emerge in the gauge formulation of the 1 + 1 dimensional gravity, where u_1 and u_2 have the meaning of Zweibein fields [10]. These equations are similar to the “fictitious” or “mirror-image” systems with *negative* friction, which appear into the thermo-field approach to the damped oscillator [11]. Anyway, the role (if any) of Eqs. (2.24) in the modeling of complex organisms remains to be elucidated.

III. LINEARIZABLE QUADRATIC MODELS

Here we shall build up RD systems which are quadratic and linearizable. We remind the reader that some of these models find applications in the study of isothermal autocatalytic chemical reactions [12]. In this context it is important the existence of propagating fronts (or travelling waves) describing the advance of the reaction. Thus, the investigation of RD systems allowing exact solutions of this kind can be a guide for the construction of models which may interpret realistic chemical situations.

Let us start from a closed prolongation algebra and study the explicit form of the compatibility condition expressed by $y_{xt} = y_{xt}$. In the non-degenerate case, this condition is Eq. (2.20), namely a set of quadratic evolution equations, one for each independent Lie algebra generator appearing after the expansion of the commutators. Since Φ and Ψ are the field dependent terms, the quadratic, linear and constant terms are respectively

$$[\Phi, [K, \Psi]], \quad [\Phi, D] + [K, [K, \Psi]], \quad [K, D]. \quad (3.1)$$

Now, let us choose a definite Lie algebra \mathcal{L} . First, we fix the set of commuting elements $H = \{H_k\}$ involved in the definition of Φ and Ψ (see Eq. (2.17c)). Second, we choose the elements K and D . Finally, we evaluate the quantities (3.1). The contributions proportional to the generators H give genuine evolution equations. All the other possible contributions fix constraints on the fields.

If we look for non trivial systems, free of constraints, then we must solve the following algebraic problem: find a Lie algebra \mathcal{L} such that: (i) \mathcal{L} has an abelian subalgebra \mathcal{A} with dimension greater than 2 and (ii) given a basis $\{H_i\}$ of \mathcal{A} there exists $K \in \mathcal{L}$ such that

$$[H_i, [H_j, K]] \in \mathcal{A}. \quad (3.2)$$

In theory, we do not know if this problem admits solutions. For instance, if we assume \mathcal{L} to be a semisimple algebra and identify \mathcal{A} with its Cartan subalgebra, then in the Cartan-Weyl basis [13] we have

$$[H_i, H_j] = 0, \quad (3.3a)$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad (3.3b)$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, \quad (3.3c)$$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i, \quad (3.3d)$$

and for a general K

$$K = \sum_i \beta_i H_i + \sum_\alpha \gamma_\alpha E_\alpha, \quad (3.4)$$

we obtain

$$[H_i, [H_j, K]] = \sum_\alpha \alpha_i \alpha_j \gamma_\alpha E_\alpha, \quad (3.5)$$

which does not belong to \mathcal{A} .

However, a simple class of algebras with the desired property is provided by the Ansatz

$$[K, D] = 0, \quad [K, \mathcal{A}] \sim D, \quad [K, D] \in \mathcal{A}. \quad (3.6)$$

Indeed, application of the Jacobi identity determines all the commutators which turn out to be

$$[H_i, H_j] = 0, \quad i, j = 1, \dots, N \quad (3.7a)$$

$$[D, K] = \gamma_i H_i, \quad (3.7b)$$

$$[K, H_i] = \mu_i D, \quad (3.7c)$$

$$[D, H_i] = \mu_i \lambda_j H_j, \quad (3.7d)$$

where γ , μ and λ are N components vectors which must satisfy

$$\mu^T \lambda = 0. \quad (3.8)$$

Now, let u stand for the column vector of the fields and

$$A = \text{diag}(\alpha_1, \dots, \alpha_N). \quad (3.9)$$

Expanding the terms in Eq. (3.1) we get

$$u_t = A u_{xx} + (\lambda \mu^T u + \gamma)(\mu^T A u + 1). \quad (3.10)$$

In the case $N = 2$ it is natural to perform a change of variables and introduce in place of u_1 and u_2 , the new fields

$$X = \mu^T u, \quad Y = \mu^T A u. \quad (3.11)$$

Exploiting the relation $\mu^T \lambda = 0$ it is straightforward to show that the two evolution equations for (X, Y) are

$$X_t = Y_{xx} + \mu^T \gamma (Y + 1), \quad (3.12a)$$

$$Y_t = (\alpha_1 + \alpha_2) Y_{xx} - \alpha_1 \alpha_2 X_{xx} + (\mu^T A \lambda X + \mu^T A \gamma)(Y + 1). \quad (3.12b)$$

It is convenient to make the shift

$$X \rightarrow X + \frac{\mu^T A \gamma}{\mu^T A \lambda}, \quad (3.13a)$$

$$Y \rightarrow Y + 1, \quad (3.13b)$$

in Eqs. (3.12). This yields

$$X_t = Y_{xx} + \mu^T \gamma Y, \quad (3.14a)$$

$$Y_t = (\alpha_1 + \alpha_2) Y_{xx} - \alpha_1 \alpha_2 X_{xx} + \beta(\alpha_1 - \alpha_2) XY, \quad (3.14b)$$

where

$$\mu^T A \lambda = \mu_1 \lambda_1 (\alpha_1 - \alpha_2) \equiv \beta(\alpha_1 - \alpha_2), \quad (3.15)$$

$\beta = \mu_1 \lambda_1$ and Eq. (3.8) has been used. Now, let us look for solutions of the travelling wave type where both X and Y depend on $\xi = x + vt$. Furthermore, by taking $\mu^T \gamma = 0$ and, consequently, $\gamma = \zeta \lambda$ (ζ being a constant factor), the shift in Eq. (3.13) becomes simply

$$X \rightarrow X + \zeta, \quad (3.16a)$$

$$Y \rightarrow Y + 1. \quad (3.16b)$$

Moreover, we can integrate the first of Eqs. (3.14) and obtain

$$X = \frac{1}{v} Y' + c_0, \quad (3.17)$$

where $Y' = dY/d\xi$ and c_0 is a constant of integration. The second equation reads

$$v Y' = (\alpha_1 + \alpha_2) Y'' - \frac{\alpha_1 \alpha_2}{v} Y''' + \frac{\beta}{v} (\alpha_1 - \alpha_2) Y Y' + \beta c_0 Y, \quad (3.18)$$

which entails

$$Y'' - v \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) Y' + \frac{v^2}{\alpha_1 \alpha_2} Y - \frac{\beta}{2\alpha_1 \alpha_2} (\alpha_1 - \alpha_2) Y^2 + c_1 = 0 \quad (3.19)$$

for $c_0 = 0$, with c_1 arbitrary constant.

When the two diffusion constants (α_1, α_2) take the special values $(\alpha, 0)$ or $(\alpha, -\alpha)$ one of the coefficients in Eq. (3.19) vanishes. Below, we shall discuss separately these singular cases and the general one.

Anyhow, independently from the value of (α_1, α_2) , we can write the following expressions for the fields u_1 and u_2 in terms of X and Y (see Eq. (3.11)) and for the relative evolution equations

$$u_1 = \frac{1}{\mu_1(\alpha_2 - \alpha_1)}(\alpha_2 X - Y), \quad (3.20a)$$

$$u_2 = \frac{\lambda_2}{\lambda_1 \mu_1(\alpha_2 - \alpha_1)}(\alpha_1 X - Y), \quad \mu_2 = -\frac{\lambda_1}{\lambda_2} \mu_1, \quad (3.20b)$$

$$u_{1t} = \alpha_1 u_{1xx} + \lambda_1 R(u_1, u_2), \quad (3.20c)$$

$$u_{2t} = \alpha_2 u_{2xx} + \lambda_2 R(u_1, u_2), \quad (3.20d)$$

$$R(u_1, u_2) = \left[\mu_1 \left(u_1 - \frac{\lambda_1}{\lambda_2} u_2 \right) + \zeta \right] \left[\mu_1 \left(\alpha_1 u_1 - \frac{\lambda_1}{\lambda_2} \alpha_2 u_2 \right) + 1 \right]. \quad (3.20e)$$

A. Case I: $\alpha_1 = \alpha$, $\alpha_2 = 0$

Equation (3.19) becomes

$$Y' = \frac{v}{\alpha} Y - \frac{\beta}{2v} Y^2, \quad (3.21)$$

which may be integrated to give

$$Y(\xi) = \frac{v^2}{\alpha\beta} \left(1 + \tanh \frac{v}{2\alpha} (\xi - \xi_0) \right). \quad (3.22)$$

Computing X from Eq. (3.17) (with $c_0 = 0$), we find

$$T = \tanh \frac{v}{2\alpha} (\xi - \xi_0), \quad (3.23a)$$

$$X = \frac{v^2}{2\alpha^2 \mu_1 \lambda_1} (1 - T^2) - \zeta, \quad (3.23b)$$

$$Y = \frac{v^2}{\alpha \mu_1 \lambda_1} (1 + T) - 1, \quad (3.23c)$$

with the help of Eq. (3.16).

B. Case II: $\alpha_1 = \alpha$, $\alpha_2 = -\alpha$

We have

$$Y'' - \frac{v^2}{\alpha^2} Y + \frac{\beta}{\alpha} Y^2 + c_1 = 0, \quad (3.24)$$

which furnishes

$$Y'^2 = k_1 + k_2 Y + \frac{v^2}{\alpha^2} Y^2 - \frac{2\beta}{3\alpha} Y^3 \quad (3.25)$$

where k_1 and k_2 are arbitrary constants. The change of variable

$$Y = -\frac{6\alpha}{\beta}\varphi + \frac{v^2}{2\alpha\beta}, \quad (3.26)$$

leads to

$$\varphi'^2 = 4\varphi^3 - g_2\varphi - g_3, \quad (3.27a)$$

$$g_2 = \frac{\beta}{6\alpha}k_2 + \frac{v^4}{12\alpha^4}, \quad (3.27b)$$

$$g_3 = -\frac{\beta^2}{36\alpha^2}k_1 - \frac{\beta v^2}{72\alpha^3}k_2 - \frac{v^6}{216\alpha^6}, \quad (3.27c)$$

and therefore

$$Y(\xi) = -\frac{6\alpha}{\beta}\mathcal{P}(\xi - \xi_0, g_2, g_3) + \frac{v^2}{2\alpha\beta} \quad (3.28)$$

where \mathcal{P} is the Weierstrass function [14] and ξ_0 an arbitrary (complex) constant. In the particular case $k_1 = k_2 = 0$, we have $\Delta = g_2^3 - 27g_3^2 = 0$ which involves elementary functions only. Using

$$\mathcal{P}(z, 12c^2, -8c^3) = c + \frac{3c}{\sinh^2(z\sqrt{3c})}, \quad (3.29)$$

we find the following expressions for the fields X, Y

$$X = -\frac{3v^2}{2\alpha^2\mu_1\lambda_1}T(1 - T^2) - \zeta, \quad (3.30a)$$

$$Y = \frac{3v^2}{2\alpha\mu_1\lambda_1}(1 - T^2) - 1, \quad (3.30b)$$

$$T = \tanh \frac{v}{2\alpha}(\xi - \xi_0). \quad (3.30c)$$

C. Case III: α_1 and α_2 arbitrary

In general, Eq. (3.19) can be written as

$$Y'' = aY' + bY + cY^2 + c_1, \quad (3.31)$$

where the coefficients a, b and c are defined by

$$a = v \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right), \quad (3.32a)$$

$$b = -\frac{v^2}{\alpha_1\alpha_2}, \quad (3.32b)$$

$$c = \frac{\beta}{2\alpha_1\alpha_2}(\alpha_1 - \alpha_2). \quad (3.32c)$$

If we introduce the new dependent variable

$$\tilde{Y} = Y - Y_0, \quad (3.33)$$

where

$$Y_0 = -\frac{1}{2c} \left(\frac{6}{25}a^2 + b \right), \quad (3.34)$$

and choose c_1 to be

$$c_1 = -bY_0 - cY_0^2, \quad (3.35)$$

then \tilde{Y} satisfies

$$\tilde{Y}'' = a\tilde{Y}' - \frac{6}{25}a^2\tilde{Y} + c\tilde{Y}^2. \quad (3.36)$$

In Appendix A, we show that the above equation affords the exact solution

$$\tilde{Y}(\xi) = \exp\left(\frac{2a}{5}\xi\right) \mathcal{P}\left(\frac{5}{a}\sqrt{\frac{c}{6}}\exp\left(\frac{a\xi}{5}\right) + k_1, 0, k_0\right), \quad (3.37)$$

(k_0 and k_1 are constants of integration) corresponding to the so-called equianharmonic case [14]. For $k_0 = 0$, we deduce

$$X = \frac{3a^3}{250cv}(1-T)(1+T)^2 - \zeta, \quad (3.38a)$$

$$Y = \frac{3a^2}{50c}(1+T)^2 - 1 - \frac{1}{2c} \left(\frac{6}{25}a^2 + b \right), \quad (3.38b)$$

$$T = \tanh \frac{a\xi}{10}, \quad (3.38c)$$

where $\mathcal{P}(z, 0, 0) = z^{-2}$ has been exploited.

D. Case IV: A particular solution for $\mu^T\gamma \neq 0$

The condition $\mu^T\gamma = 0$ has been crucial in the previous calculations because it has allowed the integration of Eq. (3.14a). In such a way, some exact solutions turn out to be polynomials in $\tanh(k\xi)$ for a certain k . In the case $\mu^T\gamma \neq 0$, solutions of the similar kind can be obtained by inserting the Ansatz

$$X = \sum_{n=0}^{N+1} a_n T^n, \quad Y = \sum_{n=0}^N b_n T^n, \quad (3.39a)$$

$$T = \tanh(k\xi), \quad (3.39b)$$

into Eqs. (3.14) (N arbitrary). The analysis of the case $N = 2$, provides the solution

$$X = \frac{3}{\rho_2 v} \left[(3D\rho_1 - v^2) \sqrt{5\rho_1 - \frac{v^2}{D}} T - D \left(5\rho_1 - \frac{v^2}{D} \right)^{3/2} T^3 \right] - \frac{\mu_1\gamma_1 - \mu_2\gamma_2}{2\mu_1\lambda_1},$$

$$Y = -\frac{3D}{\rho_2} \left(5\rho_1 - \frac{v^2}{D} \right) (1 - T^2) - 1, \quad (3.40a)$$

$$T = \tanh \left(\frac{\xi}{2} \sqrt{5\rho_1 - \frac{v^2}{D}} \right), \quad (3.40b)$$

where

$$\alpha_1 + \alpha_2 = 0, \quad D = \alpha_1 \alpha_2, \quad \rho_1 = \mu^T \gamma, \quad \rho_2 = \mu^T A \lambda. \quad (3.41)$$

In Fig. 1, we plot the fields $u_1(x, 0)$ and $u_2(x, 0)$ in the four cases and with a choice of the parameters (specified in the captions) such that reaction fronts arise.

IV. PSEUDOPOTENTIAL FORMULATION OF A QUADRATIC MODEL

In this Section, we formulate and solve the pseudopotential equations for a particular linearizable quadratic model. We show that a bootstrap structure emerges and new solutions can be obtained in terms of the old ones found by direct inspection. In doing so, let us assume

$$\lambda = (1, 1), \quad \gamma = (0, 0), \quad \mu = (1, -1), \quad (4.1)$$

in Eq. (3.10), which becomes the pair of evolution equations

$$u_{1t} = \alpha_1 u_{1xx} + R(u_1, u_2), \quad (4.2a)$$

$$u_{2t} = \alpha_2 u_{2xx} + R(u_1, u_2), \quad (4.2b)$$

with

$$R(u_1, u_2) = (u_1 - u_2)(\alpha_1 u_1 - \alpha_2 u_2 + 1). \quad (4.3)$$

In this case, the algebra (3.7) reads

$$[H_1, H_2] = [D, K] = 0, \quad (4.4a)$$

$$[K, H_1] = -[K, H_2] = D, \quad (4.4b)$$

$$[D, H_1] = -[D, H_2] = H_1 + H_2. \quad (4.4c)$$

Up to now we have always interpreted the abstract elements H_1 , H_2 , K and D as matrices belonging to a given N -dimensional linear representation of the algebra. In such a case, the evolution equations for the pseudopotential (see Eqs. (2.17) and (2.18)) take the form

$$y_{ix} = F_{ij}(u) y_j, \quad (4.5a)$$

$$y_{ix} = G_{ij}(u, u_x) y_j, \quad (4.5b)$$

where F_{ij} and G_{ij} are field dependent matrices. Hereafter, for convenience, we shall write Eqs. (4.5) in the operator form

$$\mathcal{Y}_x = \mathcal{F}, \quad (4.6a)$$

$$\mathcal{Y}_t = \mathcal{G}, \quad (4.6b)$$

where

$$\mathcal{Y} = y_i \partial_i, \quad \mathcal{F} = F_{ij}(u) y_j \partial_i, \quad \mathcal{G} = G_{ij}(u) y_j \partial_i, \quad (4.7)$$

with $\partial_i = \partial/\partial y_i$. In this way, one has associated with each abstract element H_1 , H_2 , K and D a differential operator (vector field) whose components, in the basis $\{\partial_i\}$, are linear functions of the pseudopotential variables. It is easy to see that this limitation is not necessary. Actually, all the equations go unchanged if arbitrary vector fields are considered. Taking this more general attitude, from the relation

$$[\mathcal{F}, \mathcal{G}] = \mathcal{F}\mathcal{G} - \mathcal{G}\mathcal{F} = -[F, G]_{ij}y_j\partial_i, \quad (4.8)$$

we see that the vector fields H_1 , H_2 , K and D satisfy the algebra

$$[H_1, H_2] = [D, K] = 0, \quad (4.9a)$$

$$[K, H_1] = -[K, H_2] = -D, \quad (4.9b)$$

$$[D, H_1] = -[D, H_2] = -(H_1 + H_2), \quad (4.9c)$$

which differs from Eqs. (4.4) by a change of sign in the right hand side. A possible realization of Eqs. (4.9) with a two component pseudopotential $y = (y_1, y_2)$ is

$$H_1 = \frac{1}{8}\frac{\partial}{\partial y_1} + \frac{1}{2}\frac{\partial}{\partial y_2}, \quad (4.10a)$$

$$H_2 = \frac{1}{8}\frac{\partial}{\partial y_1} - \frac{1}{2}\frac{\partial}{\partial y_2}, \quad (4.10b)$$

$$K = \frac{1}{2}y_2^2\frac{\partial}{\partial y_1}, \quad (4.10c)$$

$$D = \frac{1}{2}y_2\frac{\partial}{\partial y_1}. \quad (4.10d)$$

Hence, the pseudopotential equations (4.6) take the form

$$y_{1x} = \frac{1}{8}(u_1 + u_2) + \frac{1}{2}y_2^2, \quad (4.11a)$$

$$y_{2x} = \frac{1}{2}(u_1 - u_2), \quad (4.11b)$$

$$y_{1t} = \frac{1}{8}(\alpha_1 u_{1x} + \alpha_2 u_{2x}) + \frac{1}{2}y_2(1 + \alpha_1 u_1 - \alpha_2 u_2), \quad (4.11c)$$

$$y_{1t} = \frac{1}{2}(\alpha_1 u_{1x} - \alpha_2 u_{2x}). \quad (4.11d)$$

Equations (4.11a)-(4.11b) produce

$$u_1 = 4y_{1x} + y_{2x} - 2y_2^2, \quad (4.12a)$$

$$u_2 = 4y_{1x} - y_{2x} - 2y_2^2, \quad (4.12b)$$

which can be used to eliminate u_1 and u_2 from Eqs. (4.11c)-(4.11d). At this stage, let us seek travelling wave solutions of the quadratic model (4.2), i.e. solutions of the type $u_i = u_i(\xi)$ where $\xi = x + vt$ and $i = 1, 2$. To this aim, let us start from Eq. (4.11d), which can be written as

$$\frac{d}{d\xi} \left(-vy_1 + \frac{1}{2}\alpha_1 u_1 - \frac{1}{2}\alpha_2 u_2 \right) = 0. \quad (4.13)$$

The integration of Eq. (4.13) gives

$$y'_1(\xi) = \frac{1}{4(\alpha_1 - \alpha_2)} \left[2c_0 - 2vy_2 + 2(\alpha_2 - \alpha_1)y_2^2 + (\alpha_1 + \alpha_2)y'_2 \right] \quad (4.14)$$

with the help of Eqs. (4.12), where c_0 is a constant of integration. Taking account of Eq. (4.14), from Eq. (4.11c) we get

$$y_2'' - v \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) y_2' + v \left(\frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right) y_2^2 + y_2 \left[\frac{v^2}{\alpha_1 \alpha_2} + \left(\frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right) (2c_0 - 1) \right] - \frac{c_0 v}{\alpha_1 \alpha_2} = 0, \quad (4.15)$$

where only the pseudopotential component y_2 is involved. This equation may be solved by the procedure exploited in Case III. Precisely, we could make a shift $y \rightarrow y + y_0$ and fix the constants y_0, c_0 in such a way that Eq. (4.15) takes the form Eq. (3.36). However, for simplicity, here we adopt a different approach: we take $c_0 = 0$ and fix v by matching Eq. (3.36). Thus, we obtain

$$v = 5 \sqrt{\frac{\alpha_1 \alpha_2 (\alpha_2 - \alpha_1)}{(2\alpha_2 - 3\alpha_1)(3\alpha_2 - 2\alpha_1)}}, \quad (4.16)$$

which is real when

$$\alpha_1 < \frac{2}{3}\alpha_2, \quad \text{or} \quad \alpha_2 < \alpha_1 < \frac{3}{2}\alpha_2. \quad (4.17)$$

Just to furnish an explicit numerical example, let us consider $\alpha_1 = 2, \alpha_2 = 3/2$. Formula (4.16) gives $v = 5$ and Eq. (4.15) becomes

$$y_2''(\xi) - \frac{35}{6}y_2'(\xi) + \frac{49}{6}y_2(\xi) - \frac{5}{6}y_2^2(\xi) = 0. \quad (4.18)$$

As we have seen in Case III, a two-parameter solution of Eq. (4.18) is

$$y_2(\xi) = \exp\left(\frac{7}{3}\xi\right) \mathcal{P}\left(\frac{\sqrt{5}}{7} \exp\left(\frac{7}{6}\xi\right) + k_1, 0, k_0\right), \quad (4.19)$$

with k_0 and k_1 arbitrary constants. In particular, taking $k_0 = 0$ (see the discussion at the end of Case III) we obtain

$$y_2(\xi) = \frac{49}{20} \left(1 + \tanh\left(\frac{7}{12}(\xi - \xi_0)\right) \right)^2, \quad (4.20)$$

and, from Eqs. (4.12) and Eqs. (4.14),

$$u_1 = \frac{49}{20}(1 + T)^2(13 + 7T), \quad (4.21a)$$

$$u_2 = \frac{49}{15}(1 + T)^2(8 + 7T), \quad (4.21b)$$

$$T = \tanh\left[\frac{7}{12}(x - x_0 + 5t)\right], \quad (4.21c)$$

with x_0 arbitrary constant.

V. THE GIERER-MEINHARDT MODELS

In this Section we apply the prolongation analysis to the RD models of the Gierer-Meinhardt (GM) type. In the biological context, of a special interest is the cubic GM system, which describes the interplay between an *activator* field $u(x, t)$ and its *substrate* counterpart $w(x, t)$. This model is defined by the two equations

$$u_t = \alpha u_{xx} + R_1, \quad (5.1a)$$

$$w_t = \beta w_{xx} + R_2, \quad (5.1b)$$

with

$$R_1 = \epsilon(u^2 w - u), \quad (5.2a)$$

$$R_2 = \lambda(1 - u^2 w). \quad (5.2b)$$

The two fields diffuse with diffusion coefficients which are generally different. The concentration of the field u decays according to the $-\epsilon u$ term, but is enhanced by the substrate w via the production term $\epsilon u^2 w$. On the other hand, the substrate is injected in the domain of the reaction with a constant rate λ while its depletion is controlled by the same nonlinear reaction term.

The two opposite fixed points are

$$\begin{aligned} u &= 1, & w &= 1, \\ u &= 0, & w &= \lambda t + \omega, & \omega_t &= \beta \omega_{xx}. \end{aligned} \quad (5.3)$$

The former describes a scenario in which the activator field has reached a balance between catalyzation by substrate and substrate depletion. In the latter case the activator field is absent and the substrate arises due to the injection constant term with inhomogeneities damped in time according to the heat equation.

Moreover, according to the general analysis, it could be interesting also to investigate the model obtained by replacing in Eqs. (5.2) the cubic interaction $u^2 w$ with the quadratic term uw . We shall refer to this system as the quadratic Gierer-Meinhardt model.

A. The cubic GM model

Let us start from the prolongation equations

$$y_x = F(u, w, y), \quad (5.4a)$$

$$y_t = G(u, u_x, w, w_x, y), \quad (5.4b)$$

whose compatibility condition provides

$$\alpha F_u = G_{u_x}, \quad (5.5a)$$

$$\beta F_w = G_{w_x}, \quad (5.5b)$$

$$F_u R_1 + F_w R_2 + [F, G] = G_u u_x + G_w w_x. \quad (5.5c)$$

From these equations one finds that $G_{u_x u_x} = G_{w_x w_x} = 0$ and, therefore,

$$G = A(u, w, y)u_x + B(u, w, y)w_x + C(u, w, y), \quad (5.6)$$

where A , B and C are vector fields of integration. Putting back this result into Eq. (5.5c) and equating to zero the coefficients of the independent monomials 1 , u_x , w_x , $u_x w_x$, u_x^2 , w_x^2 , we obtain

$$A_u = 0, \quad (5.7a)$$

$$B_w = 0, \quad (5.7b)$$

$$A_w + B_u = 0, \quad (5.7c)$$

$$[F, A] = C_u, \quad (5.7d)$$

$$[F, B] = C_w, \quad (5.7e)$$

$$F_u R_1 + F_w R_2 + [F, C] = 0, \quad (5.7f)$$

from which

$$A = a_0(y)w + a_1(y), \quad (5.8a)$$

$$B = -a_0(y)u + a_2(y), \quad (5.8b)$$

a_0 , a_1 and a_2 being vector fields of integration. Moreover, Eqs. (5.5a-5.5b) imply

$$\begin{cases} \alpha F_u = G_{u_x}, \\ \beta F_w = G_{w_x}, \end{cases} \Rightarrow \begin{cases} F = \frac{1}{\alpha}(a_0 u w + a_1 u + h(w, y)), \\ \frac{\beta}{\alpha}(a_0 u + h_w) = -a_0 u + a_2. \end{cases} \quad (5.9)$$

Now, two cases have to be distinguished: if i) $\alpha + \beta \neq 0$, then

$$a_0 = 0, \quad (5.10a)$$

$$h = \frac{\alpha}{\beta} a_2 w + a_3, \quad (5.10b)$$

while if ii) $\alpha + \beta = 0$, a_0 may be different from zero and

$$h = -a_2 w + a_3. \quad (5.11)$$

First let us analyze the non-degenerate case i) which is the one relevant to proper RD systems [1].

1. The non-degenerate case: $\alpha + \beta \neq 0$

The equations to be satisfied are (see Eqs. (5.7))

$$[F, A] = C_u, \quad (5.12a)$$

$$[F, B] = C_w, \quad (5.12b)$$

$$F_u R_1 + F_w R_2 + [F, C] = 0, \quad (5.12c)$$

where

$$A = a_1, \quad B = a_2, \quad (5.13a)$$

$$F = \frac{1}{\alpha} \left(a_1 u + \frac{\alpha}{\beta} a_2 w + a_3 \right), \quad (5.13b)$$

$$G = a_1 u_x + a_2 w_x + C. \quad (5.13c)$$

Requiring that $C_{uw} = C_{wu}$ (see Eqs.(5.12a-5.12b)), we find

$$[a_1, a_2] = 0, \quad (5.14)$$

and

$$C = -\frac{1}{\alpha} [a_1, a_3] u - \frac{1}{\alpha} [a_2, a_3] w + a_4. \quad (5.15)$$

Expanding Eq. (5.12c) and collecting the coefficients of the monomials 1, u , w , u^2 , w^2 , uw , $u^2 w$ we are led to the following incomplete algebra

$$a_2 = \frac{\beta \epsilon}{\alpha \lambda} a_1, \quad (5.16a)$$

$$[a_1, [a_1, a_3]] = 0, \quad (5.16b)$$

$$[a_3, a_4] = -\epsilon a_1, \quad (5.16c)$$

$$[a_1, a_4] = \frac{1}{\alpha} [a_3, [a_1, a_3]] + \epsilon a_1, \quad (5.16d)$$

$$[a_1, a_4] = \frac{\beta}{\alpha^2} [a_3, [a_1, a_3]]. \quad (5.16e)$$

This algebra corresponds to Eqs. (2.19). The linear dependence between a_1 and a_2 is due to the cubic terms appearing in the evolution equations (5.1). The compatibility condition $y_{tx} = y_{xt}$ together with the information encoded into the incomplete algebra gives

$$a_1 \{\text{first evolution eq.}\} \sim a_2 \{\text{second evolution eq.}\}. \quad (5.17)$$

On the other hand, since a_1 and a_2 are linearly dependent, this equation is not equivalent to the pair of evolution equations (5.1). However, some non trivial structure remains because the cubic terms in R_1 and R_2 are the same monomial in the fields. If this had not been the case, then a_1 and a_2 would have been zero with a corresponding trivial algebraic structure.

In order to close the above incomplete algebra, let us set

$$[a_1, a_3] = a_5. \quad (5.18)$$

If we introduce the parameter

$$\xi^2 = \frac{\beta - \alpha}{\epsilon \alpha^2}, \quad (5.19)$$

and rescale the generators as follows

$$a_1 = -\frac{\beta\epsilon\alpha^2}{(\beta-\alpha)^2}\xi A_1, \quad (5.20a)$$

$$a_3 = \frac{1}{\xi}A_3, \quad (5.20b)$$

$$a_4 = \frac{\epsilon\beta}{\beta-\alpha}A_4, \quad (5.20c)$$

$$a_5 = -\frac{\beta\epsilon\alpha^2}{(\beta-\alpha)^2}A_5, \quad (5.20d)$$

we have the commutation relations

$$[A_1, A_3] = A_5, \quad [A_1, A_4] = A_1, \quad (5.21a)$$

$$[A_1, A_5] = 0, \quad [A_3, A_4] = A_1, \quad (5.21b)$$

$$[A_3, A_5] = A_1, \quad [A_5, A_4] = A_5, \quad (5.21c)$$

in terms of $\{A_i\}$ $i = 1, \dots, 5$. Equations (5.21) define a closed Lie algebra as one can check by computing all the non trivial cases via the Jacobi identity. Moreover, if we put

$$A_6 = A_4 - A_5, \quad (5.22)$$

then Eqs. (5.21) become

$$[A_1, A_5] = 0, \quad [A_3, A_6] = 0, \quad (5.23a)$$

$$[A_1, A_3] = A_5, \quad [A_1, A_6] = A_1, \quad (5.23b)$$

$$[A_3, A_5] = A_1, \quad [X, A_6] = A_5, \quad (5.23c)$$

where we recognize the Lie algebra $sim(2)$ of the similitude group in the plane [2]. The operators A_1 and A_5 are the generators of two orthogonal translations, A_3 is the generator of rotations in the plane and, finally, A_6 is the generator of isotropic scalings. This geometrical interpretation gives also the basic vector field realization of the algebra, i.e.

$$A_1 = \frac{\partial}{\partial x}, \quad A_5 = \frac{\partial}{\partial y}, \quad (5.24a)$$

$$A_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad (5.24b)$$

$$A_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad (5.24c)$$

It is rather intriguing the fact that the above rescaling requires $\alpha \neq \beta$ since this is also a necessary condition for Turing instability to occur in these kind of systems. The point $\alpha = \beta$ is singular, in the sense that the incomplete algebra (2.19) collapses to

$$a_1 = 0, \quad [a_3, a_4] = 0, \quad (5.25)$$

which implies the trivial result

$$y_x = \frac{1}{\alpha}a_3, \quad (5.26a)$$

$$y_t = a_4, \quad (5.26b)$$

$$[a_3, a_4] = 0. \quad (5.26c)$$

2. The degenerate case: $\alpha + \beta = 0$

From Eqs. (5.7) we deduce

$$F = a_0 uw + a_1 u - a_2 w + a_3, \quad (5.27a)$$

$$G = Au_x + Bw_x + C, \quad (5.27b)$$

$$A = \alpha(a_0 w + a_1), \quad (5.27c)$$

$$B = \alpha(-a_0 u + a_2), \quad (5.27d)$$

$$C = \alpha \{uw [a_1, a_2] - u [a_1, a_3] - w [a_2, a_3] + a_4\}. \quad (5.27e)$$

We observe that in Eq. (5.12c) the coefficient of the monomial $u^3 w$ must vanish and this gives $a_0 = 0$. Now, by performing the scaling $\epsilon \rightarrow \alpha\epsilon$, $\lambda \rightarrow \alpha\lambda$, the resulting prolongation algebra is

$$[a_2, [a_2, a_1]] = 0, \quad (5.28a)$$

$$[a_2, [a_2, a_3]] = 0, \quad (5.28b)$$

$$[a_1, [a_1, a_3]] = 0, \quad (5.28c)$$

$$[a_3, [a_1, a_2]] = 0, \quad (5.28d)$$

$$[a_1, a_4] = [a_3, [a_1, a_3]] + \epsilon a_1, \quad (5.28e)$$

$$[a_2, a_4] = [a_3, [a_3, a_2]], \quad (5.28f)$$

$$[a_3, a_4] = \lambda a_2, \quad (5.28g)$$

$$[a_1, [a_1, a_2]] = -\epsilon a_1 - \lambda a_2. \quad (5.28h)$$

Here, in theory, the generators a_1 and a_2 are not constrained to be linearly dependent. However, complete linearization requires that specific algebraic constraints have to be satisfied. For a generic model, some mechanism must ruin the equivalence between the evolution equations and the pseudopotential formulation. In this case, as we promptly show, a linear dependence arises among the generators.

From Eq. (5.28h) we find

$$\epsilon [a_1, a_2] = [a_2, [a_1, [a_1, a_2]]] = \quad (5.29a)$$

$$= [a_1, [a_2, [a_1, a_2]]] - [[a_1, a_2], [a_2, a_1]] = 0, \quad (5.29b)$$

hence, still from Eq. (5.28h),

$$a_2 = -\frac{\epsilon}{\lambda} a_1, \quad (5.30)$$

and the incomplete algebra turns out to close on the algebra $\mathfrak{sim}(2)$.

B. The quadratic GM model

In a similar way, also the quadratic Gierer-Meinhardt model turns out to be non linearizable. All the details of the analogous computations are contained in Appendix B.

VI. SPECIAL SOLUTIONS OF THE NON-DEGENERATE CUBIC GM MODEL

In this Section we focus on the cubic GM model in the non-degenerate case and discuss the consequences of the pseudopotential formulation, which can be written as

$$y_x = Fy = - \left(u + \frac{\epsilon}{\lambda} w \right) \mu \xi A_1 y + \frac{1}{\alpha \xi} A_3 y, \quad (6.1a)$$

$$y_t = Gy = - \left(u + \frac{\beta \epsilon}{\alpha \lambda} w \right) \alpha \mu \xi A_1 y + \left(u + \frac{\beta \epsilon}{\alpha \lambda} w \right) \mu A_5 y + \nu A_4 y, \quad (6.1b)$$

where

$$\mu = \frac{\beta \epsilon \alpha}{(\beta - \alpha)^2}, \quad \nu = \frac{\epsilon \beta}{\beta - \alpha}, \quad \xi^2 = \frac{\beta - \alpha}{\epsilon \alpha^2}. \quad (6.2)$$

An explicit representation of the Lie algebra $sim(2)$ is obtained from the geometric transformations of the similitude group and is given by

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (6.3)$$

$$A_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (6.4)$$

Since the third row is null, the third component of the pseudopotential is constant, $y_3 = c$. The other equations turn out to be

$$y_1 = -\alpha \xi y_{2x}, \quad (6.5a)$$

$$y_{2xx} + \frac{1}{(\alpha \xi)^2} y_2 = \frac{\mu c}{\alpha} \left(u + \frac{\epsilon}{\lambda} w \right) - \frac{c}{(\alpha \xi)^2}, \quad (6.5b)$$

$$y_{2t} = \mu c \left(u + \frac{\epsilon \beta}{\lambda \alpha} w \right) - \nu y_2. \quad (6.5c)$$

The compatibility condition $y_{2,txx} = y_{2,txt}$ gives the equation

$$\partial_t \left(u + \frac{\epsilon}{\lambda} w \right) = \alpha \partial_x^2 \left(u + \frac{\epsilon \beta}{\lambda \alpha} w \right) + \epsilon(1 - u). \quad (6.6)$$

If we write u and w in terms of the pseudopotential $y \equiv y_2$ we find

$$u = 1 + \frac{\beta - \alpha}{\epsilon \beta} \frac{1}{c} (\beta D_x - D_t) y, \quad (6.7a)$$

$$w = -\frac{\lambda \alpha}{\epsilon \beta} \left[1 + \frac{\beta - \alpha}{\epsilon \alpha} \frac{1}{c} (\alpha D_x - D_t) y \right], \quad (6.7b)$$

where

$$D_x = \partial_x^2 + \frac{1}{(\alpha\xi)^2}, \quad (6.8a)$$

$$D_t = \partial_t + \nu. \quad (6.8b)$$

In order to determine y , we must put these expressions into one of the genuinely nonlinear evolution equations. When this is done, one obtains

$$\Omega(\alpha)\Omega(\beta)y = \frac{\epsilon^2\beta}{\beta-\alpha} \left\{ 1 + \frac{\alpha\lambda}{\beta\epsilon} \left(1 + \frac{\beta-\alpha}{\epsilon\beta} \Omega(\beta)y \right)^2 \left(1 + \frac{\beta-\alpha}{\epsilon\alpha} \Omega(\alpha)y \right) \right\}, \quad (6.9)$$

where $\Omega(\eta) = \eta D_x - D_t$ and in particular $\Omega(\beta) = \beta \partial_x^2 - \partial_t$.

In Appendix C we discuss the role of the Casimir operator $\mathcal{C} = A_1^2 + A_5^2$ of the Euclidean subalgebra of $sim(2)$. In particular, we show that if (u, w) is a solution of Eqs. (5.1) and y a solution of Eqs. (6.1), then the new pseudopotential

$$y \rightarrow e^{-2\nu t} \mathcal{C}y \quad (6.10)$$

is a new solution.

Equation (6.9) will be applied below to find particular solutions (homogeneous and of the travelling wave type) to the cubic GM model.

A. Homogeneous solutions

Let us look for a class of solutions to Eqs. (5.1) ($\alpha \neq \beta$) assuming that

$$u + \frac{\epsilon}{\lambda}w = \phi(t), \quad (6.11)$$

where $\phi(t)$ denotes a given function of the time only. Then, from Eqs. (6.5) we get

$$u + \frac{\epsilon\beta}{\alpha\lambda}w = \alpha\xi^2(\dot{\phi} + \nu\phi) - \frac{\nu}{\mu} = \dots = 1 - \frac{\beta}{\alpha} + \frac{\beta}{\alpha}\phi + \frac{\dot{\phi}}{\epsilon} \left(\frac{\beta}{\alpha} - 1 \right), \quad (6.12)$$

and therefore

$$u = 1 - \frac{1}{\epsilon}\dot{\phi}, \quad (6.13a)$$

$$w = \frac{\lambda}{\epsilon} \left(-1 + \phi + \frac{1}{\epsilon}\dot{\phi} \right), \quad (6.13b)$$

where $\dot{\phi} = d\phi/dt$. The equations of motion (5.1) may be written as

$$u_t + \frac{\epsilon}{\lambda}w_t = \epsilon(1 - u), \quad (6.14a)$$

$$w_t = \lambda(1 - u^2w). \quad (6.14b)$$

The former is automatically satisfied by the Ansatz (6.11), while the latter may be written as

$$\phi'' = (1 - \phi') \left(1 + \frac{\lambda}{\epsilon} (1 - \phi') (1 - \phi - \phi') \right), \quad (6.15)$$

where $\phi' = d\phi/d\tau$, with $\tau = \epsilon t$. The (trivial) solutions corresponding to $u = 1$ and $u = 0$ are respectively

$$\phi = 1 + \frac{\epsilon}{\lambda}, \quad \phi = \tau - \tau_0. \quad (6.16)$$

The problem to be solved is

$$\phi'' = (1 - \phi') (1 + a(1 - \phi')(1 - \phi - \phi')), \quad a = \frac{\lambda}{\epsilon}, \quad (6.17)$$

with the initial conditions

$$\phi'(0) = 1 - u(0), \quad (6.18a)$$

$$\phi(0) = u(0) + \frac{\epsilon}{\lambda} w(0). \quad (6.18b)$$

Equation (6.17) can be elaborated by using the hodographic transformation

$$\phi' = \frac{1}{1 + \theta(\phi)}. \quad (6.19)$$

In doing so, we obtain

$$\theta' = -\theta + \theta^2(a\phi - 2) + \theta^3(a\phi - 1 - a), \quad (6.20)$$

which gives for $a = 1$

$$\theta' = \theta(\xi\theta(\theta + 1) - 1), \quad (6.21)$$

where $\theta' = d\theta/d\xi$, $\xi = \phi - 2$ and

$$\phi'(\tau) = \frac{1}{1 + \theta(\phi - 2)}. \quad (6.22)$$

It is interesting to consider the qualitative behavior of Eq. (6.21) with $\theta(0) = \alpha > 0$. This evolution problem

$$\theta'(\xi) = -\theta + \xi\theta^2 + \xi\theta^3, \quad (6.23a)$$

$$\theta(0) = \alpha > 0, \quad (6.23b)$$

has solutions which are decreasing until they eventually meet the nullcline Γ given by $\theta(\theta + 1)\xi = 1$. It seems reasonable to predict that all the solutions starting with α below some critical value α^* decay exponentially at infinity whereas all the other solutions meet Γ and thereafter explode in a finite time. This scenario, described in Fig. 2, is confirmed by the exact solution of the evolution problem when only one of the nonlinear terms is present, namely

$$\theta' = -\theta + \xi\theta^2 \Rightarrow \theta = \left(1 + \xi + (1/\theta_0 - 1)e^\xi\right)^{-1}, \quad (6.24a)$$

$$\theta' = -\theta + \xi\theta^3 \Rightarrow \theta = \left(1 + 2\xi + (1/\theta_0^2 - 1)e^{2\xi}\right)^{-1/2}. \quad (6.24b)$$

A heuristic evaluation of α^* is shown in Appendix D. Here we describe a notable approximate solution to the evolution problem (6.21). To this aim let us consider the equation

$$\xi\xi' - \frac{1}{\theta(\theta+1)}\xi' - \frac{1}{\theta^2(\theta+1)} = 0, \quad (6.25)$$

with $\xi' = d\xi/d\theta$. Now we remind the reader that the class of integrable equations

$$\xi\xi' + f(\theta)\xi' \pm f'(\theta) = 0, \quad (6.26)$$

allows the general integral

$$I = e^{\pm\xi}(\xi + f(\theta) \mp 1). \quad (6.27)$$

Actually, Eq. (6.26) can be associated with the evolution equation

$$\theta' = \mp \frac{1}{f'(\theta)}(\xi + f(\theta)), \quad (6.28)$$

or

$$\frac{df}{d\xi} = \mp(\xi + f), \quad (6.29)$$

which shows how integrability of Eq. (6.26) is just linearity in disguise. Anyhow, if we take

$$f(\theta) = -\frac{1}{\theta(\theta+1)}, \quad (6.30)$$

and the minus sign in Eq. (6.28), then we find

$$\theta' = \frac{\theta(\theta+1)}{2\theta+1} [\xi\theta(\theta+1) - 1]. \quad (6.31)$$

This is a modified equation but with a modifying factor $\frac{\theta(\theta+1)}{2\theta+1}$ which is bounded between 1/2 and 1 when $\theta > 0$ and which, therefore, may be expected to give minor changes in the solution. Another way of emphasizing the extra terms is that of writing

$$\frac{d}{d\xi} [2\theta - \log(1+\theta)] = \frac{d}{d\xi} \left[\theta + \frac{\theta^2}{2} + \dots \right] = \theta[\xi\theta(\theta+1) - 1]. \quad (6.32)$$

The integral of Eq. (6.31) is (see Eq. (6.27)):

$$I = e^{-\xi} \left[\xi - \frac{1}{\theta(\theta+1)} + 1 \right]. \quad (6.33)$$

In terms of $\theta_0 = \theta(0) > 0$ we have

$$I = 1 - \frac{1}{\theta_0(\theta_0 + 1)}, \quad (6.34)$$

and

$$\theta(\xi) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{1 + \xi - Ie^\xi}}. \quad (6.35)$$

Some curves are displayed in Fig. 3. The constant I is in the range $(-\infty, 1)$, all the solutions with $I < 0$ are regular for $\xi > 0$ and decay exponentially at $\xi \rightarrow +\infty$. The particular value $I = 0$ corresponds to the separatrix between regular and singular solutions and starts at the golden ratio $\theta_0 = (\sqrt{5} - 1)/2$. We remark that Eq. (6.22) predicts singular solutions to be associated with periodic oscillating solutions approaching limiting closed curves in the (u, w) plane as $t \rightarrow +\infty$.

In the special case of the separatrix, all the remaining integrations may be performed. At $I = 0$ we have

$$\theta(\xi) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{1 + \xi}} \rightarrow \theta(\phi - 2) = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\phi + 3}{\phi - 1}}, \quad (6.36)$$

and hence integrating Eq. (6.22) we find

$$\frac{1}{2}\phi + \frac{1}{2}(\phi - 1)\sqrt{\phi + 3}\phi - 1 + \log \left[\phi + 1 + (\phi - 1)\sqrt{\frac{\phi + 3}{\phi - 1}} \right] = \tau - \tau_0. \quad (6.37)$$

Moreover, from Eqs. (6.13)

$$u(\tau) = 1 - \phi' = \dots = \frac{\sqrt{(\phi + 3)/(\phi - 1)} - 1}{\sqrt{(\phi + 3)/(\phi - 1)} + 1}, \quad (6.38)$$

that is

$$\phi = \frac{1}{u} + u - 1. \quad (6.39)$$

Finally, substitution from Eq. (6.39) into Eq. (6.37) gives

$$-\frac{1}{2} + \frac{1}{u} + \log \frac{2}{u} = \tau - \tau_0. \quad (6.40)$$

We remark that Eq. (6.40) can be exactly solved in terms of the Lambert W function [15] which obeys the Schröder equation [16]

$$W(x) + e^{W(x)} = x. \quad (6.41)$$

Indeed, we have

$$u = \left[W \left(\frac{1}{2} e^{\frac{1}{2} + \tau - \tau_0} \right) \right]^{-1}. \quad (6.42)$$

The function w can be determined by Eqs. (6.13) with the help of Eq. (6.39), i.e.

$$w = \frac{1}{u} - 1 = W\left(\frac{1}{2}e^{\frac{1}{2}+\tau-\tau_0}\right) - 1. \quad (6.43)$$

In Fig. 4 we display the behavior of this special solution. To conclude this Section, we point out that the asymptotic behavior of $W(x)$ for large x is $W(x) \sim \log x$ and therefore in this limit

$$u(\tau) \sim \left(\tau - \tau_0 + \frac{1}{2} - \log 2\right)^{-1}, \quad (6.44a)$$

$$w(\tau) \sim \tau - \tau_0 - \frac{1}{2} - \log 2. \quad (6.44b)$$

We remark that this solution has been obtained from the approximate equation (6.31); however, it solves asymptotically the exact equations (5.1) because the extra terms in Eq. (6.32) vanish in the limit $u \rightarrow 0$ (see Eqs. (6.13) and (6.19)).

B. Travelling waves

In order to look for solutions of Eq. (6.9) of the travelling wave type, we assume that the pseudopotential y is a function of the reduced variable $\xi = x + vt$, where v is a constant. In terms of ξ we have

$$\Omega(\alpha) = \beta D^2 - vD, \quad (6.45a)$$

$$\Omega(\beta) = \alpha D^2 - vD - \epsilon, \quad (6.45b)$$

with $D = \frac{d}{d\xi}$. This formalism suggests the particular speed

$$v = \pm \beta \sqrt{\frac{\epsilon}{\alpha - \beta}}, \quad (6.46)$$

as a critical value. Actually, in this case we have the factorization

$$\Omega(\alpha) = \alpha \left(D \pm \frac{\sqrt{\epsilon(\alpha - \beta)}}{\alpha} \right) \left(D \mp \sqrt{\frac{\epsilon}{\alpha - \beta}} \right), \quad (6.47a)$$

$$\Omega(\beta) = \beta D \left(D \mp \sqrt{\frac{\epsilon}{\alpha - \beta}} \right), \quad (6.47b)$$

with a common factor. Therefore, we can set

$$\varphi = \left(D \mp \sqrt{\frac{\epsilon}{\alpha - \beta}} \right) y, \quad (6.48)$$

and get a reduced equation in terms of the variable φ . Without loss of generality, let us focus on the particular set of constants

$$\alpha = 2, \quad \beta = 1, \quad \epsilon = 1. \quad (6.49)$$

Then, $v = \pm 1$ and the relation between φ and y becomes $\varphi = (D \mp 1)y$. Equation (6.9) may be written as

$$(D \pm 1/2) (D \mp 1) D\varphi = -\frac{1}{2} \left[1 - 2\lambda (D\varphi - 1)^2 ((D \pm 1/2)\varphi - 1) \right], \quad (6.50)$$

and the fields u, w are expressed in terms of φ by

$$u = 1 - D\varphi, \quad (6.51a)$$

$$w = -2\lambda (1 - (D \pm 1/2)\varphi). \quad (6.51b)$$

The fixed points of the cubic Gierer-Meinhardt model (see Eq. (5.3)) correspond to the two exact (trivial) solutions

$$\varphi = \frac{1 + 2\lambda}{\lambda}, \quad \varphi = \xi + \text{const.} \quad (6.52)$$

It is very interesting to look for a particular phenomenological meaning of the critical value expressed by Eq. (6.46). Indeed, our numerical simulations show that something special happens at that point (see Sec. VII).

Apart from the critical speed, the general problem of finding a travelling wave solution to Eq. (6.9) may be studied by looking for interpolating solutions which at $\xi \rightarrow \pm\infty$ approach the two fixed points (6.52). This is a difficult eigenvalue problem for a 4th order nonlinear ordinary differential equation. We shall exhibit semi-analytical approximations of these interpolating solutions in Sec. VII, devoted mainly to numerical results.

VII. NUMERICAL SIMULATIONS

In this Section we discretize the evolution equations of the cubic GM model and study them from a numerical point of view. This allows for a check of our analytical predictions and makes a step forward from our approximate solutions toward the unknown exact ones and their phenomenology.

A. Homogeneous solutions

Let us consider Eqs. (5.1) and (5.2) in the case $\lambda = \epsilon$ ($a = 1$). By rescaling the time variable we can set

$$\lambda = \epsilon = 1. \quad (7.1)$$

The couple of differential equations to be integrated numerically is

$$\dot{u}(t) = u^2 w - u, \quad (7.2a)$$

$$\dot{w}(t) = 1 - u^2 w. \quad (7.2b)$$

The initial values $u(0), w(0)$ are related by

$$w(0) = 2 - u(0). \quad (7.3)$$

Since the theoretical analysis carried out in Sec. VIA shows that oscillating trajectories emerge for

$$u(0) > \frac{\alpha^*}{1 + \alpha^*}, \quad (7.4)$$

where α^* is a critical value determined in Appendix D, we shall start with $u(0)$ slightly above $\alpha^*/(1 + \alpha^*)$. Using the following second order algorithm for the integration of equations $\dot{x}_i = f_i(x)$:

$$y_i^{(n)} = x_i^{(n)} + \frac{\delta}{2} f_i(x^{(n)}), \quad (7.5a)$$

$$x_i^{(n+1)} = x_i^{(n)} + \delta f_i(y^{(n)}), \quad (7.5b)$$

with δ the discretization time step, we obtain the result pictured in Fig. 5.

B. Travelling waves

The search for travelling wave solutions of the cubic GM model amounts to the solution of a 4th order nonlinear ordinary differential equation (a reduction of Eq. (6.9)) with given boundary conditions corresponding to the two fixed points (5.3). From a biological point of view, these solutions interpolate between regimes dominated by the activator field or the substrate field and are characterized by a reaction front which separated the two regions. For a 2nd order equation, phase-space techniques permit geometrical proofs of the existence of such solutions. Here, dealing with a 4th order equation we shall build up approximate solutions and adopt them as particular initial conditions in the numerical integration.

At the critical speed, the order is only 3 and we start from this case. The equation to be solved is Eq. (6.50). Without loss of generality, let us set $\lambda = 1$. The two fixed points (6.52) are

$$\varphi = \xi + \text{constant}, \quad \varphi = 3. \quad (7.6)$$

As we shall see, the above constant can be set to zero. Let us look for a perturbed solution around $\varphi = \xi$ by writing

$$\varphi = \xi + \delta, \quad (7.7)$$

and expanding Eq. (6.50) at the first order in δ . We find

$$2\delta''' - \delta'' - \delta' = 0, \quad (7.8)$$

whose characteristic roots are 0, 1, $-1/2$. The same computation starting from

$$\varphi = 3 + \delta, \quad (7.9)$$

gives

$$2\delta''' - \delta'' - \delta' - \delta = 0, \quad (7.10)$$

which has one characteristic root on the right complex half-plane and the other two on the left half-plane. A sensible perturbation must vanish at infinity. In order to have the greatest number of free constants we match the first fixed point at $\xi \rightarrow -\infty$ and the other at $\xi \rightarrow +\infty$. Hence, we construct our approximate solution as

$$\varphi(\xi) = \begin{cases} \varphi_-(\xi) & \xi < 0, \\ \varphi_+(\xi) & \xi \geq 0, \end{cases} \quad (7.11)$$

with

$$\varphi_-(\xi) = \xi + c_1 + c_2 e^\xi, \quad (7.12a)$$

$$\varphi_+(\xi) = (c_3 \sin \nu \xi + c_4 \cos \nu \xi) e^{-\mu \xi} + 3, \quad (7.12b)$$

$$\mu = 0.366876, \quad (7.12c)$$

$$\nu = 0.520259. \quad (7.12d)$$

The constant c_1 may be eliminated and set to zero by a translation in ξ . The free constants c_2 , c_3 and c_4 are determined (rather arbitrarily) by imposing the regularity condition

$$\varphi_-^{(n)}(0) = \varphi_+^{(n)}(0), \quad n = 0, 1, 2. \quad (7.13)$$

The result is

$$c_2 = 0.225361, \quad (7.14a)$$

$$c_3 = 0.398672, \quad (7.14b)$$

$$c_4 = -2.77464, \quad (7.14c)$$

which gives an approximate solution as regular as possible and asymptotically exact.

This procedure may be extended to the more general case of an arbitrary speed. Since our interest is in the methodological viewpoint and in giving specific examples we keep the above parameters ($\alpha = 2, \beta = \lambda = \epsilon = 1$) but do not fix the speed. With these values, the travelling wave solutions of Eq. (6.9) obey

$$2y^{iv} - 3vy''' + (v^2 - 1)y'' + vy' = \quad (7.15a)$$

$$= -1 + (y'' - vy' - 1)^2(2y'' - vy' - y - 2), \quad (7.15b)$$

and the fields are expressed by

$$u = 1 + vy' - y'', \quad (7.16a)$$

$$w = -2 - y - vy' + 2y''. \quad (7.16b)$$

The two fixed points correspond to the solutions

$$y = -3, \quad y = -\frac{\xi}{v}. \quad (7.17)$$

Perturbing around the first by setting $y = -3 + \delta$ and expanding at the first order in δ we obtain a linear differential equation with constant coefficients and characteristic polynomial

$$2p^4 - 3p^3v + (v^2 - 1)p^2 + 1 = 0. \quad (7.18)$$

In Appendix E we show that for each value of the speed v the above equation has always two zeroes on the left complex half-plane and the other two on the right. Perturbing the second solution by setting $y = -\xi/v + \delta$ we find

$$p(p - v)(2p^2 - pv - 1) = 0, \quad \rightarrow p = 0, v, \frac{v \pm \sqrt{v^2 + 8}}{4}, \quad (7.19)$$

and, for the same reasons as before, we are forced to match this boundary condition on the left at $\xi \rightarrow -\infty$. Just to give a numerical result, the above procedure in the case $v = 2$ provides the following approximate travelling wave

$$y(\xi) = \begin{cases} y_-(\xi) & \xi < 0, \\ y_+(\xi) & \xi \geq 0, \end{cases} \quad (7.20)$$

with

$$y_- = c_1 e^{2\xi} + c_2 e^{\xi(1+\sqrt{3})/2} - \frac{\xi}{2}, \quad (7.21a)$$

$$y_+ = -3 + (c_3 \cos \nu \xi + c_4 \sin \nu \xi) e^{-\mu \xi}, \quad (7.21b)$$

where regularity up to the third order requires

$$c_1 = 0.255476, \quad (7.22a)$$

$$c_2 = -0.690277, \quad (7.22b)$$

$$c_3 = -1.217923, \quad (7.22c)$$

$$c_4 = 2.565199. \quad (7.22d)$$

Turning to the numerical simulations, we fixed all the constants at the values $\alpha = 2.0$, $\beta = 1.0$, $\epsilon = \lambda = 1.0$. Assuming the equations of motion to be stabilized by the diffusion terms, we have discretized them by a first order Euler scheme.

Concerning boundary conditions, they are easily dealt with at least in the case of travelling wave solutions. For generic positive speed v , the exact u wave has asymptotically constant values whereas the exact w wave tends to a constant on the right and to a linear function to the left. Using these informations we may impose the correct boundary conditions minimizing finite size effects. Anyway, we also checked independence of the results from the boundary conditions, at least when time is enough small for the size effects not to be relevant. For instance, if one exploits periodic boundary conditions and starts with the approximate (infinite volume) solution, then a perturbation is seen to arise at the boundary and the simulation must be stopped when it collides with the localized reaction front of the u wave.

Now, let us examine the output of the numerical simulation. In Fig. 6 we display the evolution of the approximate wave with $v = 2.0$. As one can see, it rapidly settles down to a stable wave travelling with the desired speed. In Fig. 7 we repeat the simulation starting from the approximate wave with speed $v = 1.0$. In this case the approximate wave evolves toward an apparently stable travelling wave with internal oscillations. In Fig. 8, where $v = 0.5$ is subcritical the same situation occurs.

These results suggest the following two scenarios below the critical wave speed: (i) there are no travelling waves or they are unstable; (ii) a stable travelling wave exists, but our approximate solutions are too crude and their evolution does not converge to it. Actually, in literature, we find examples of systems of this kind which exhibit the scenario (i) [3].

VIII. CONCLUSIONS

The results achieved in this paper show that the prolongation technique reveals a fruitful tool to investigate morphogenesis and autocatalysis modeling equations.

We have dealt with a class of RD systems including reaction terms which are both quadratic and cubic in the fields. This class comprises well-known RD equations such as the Gierer-Meinhardt models. The prolongation approach allowed us to discover the existence of an algebraic structure inherent in this kind of models. It turns out to be the algebra associated with the similitude group. This feature could be important within a programme to settle up a systematic of RD models, which might be classified according to their algebraic properties.

The comparison between special solutions drawn from theory and the corresponding ones obtained via numerical simulations is satisfactory. This represents an encouragement to extend the algebraic strategy to handle more complicated RD nonlinear evolution equations, keeping in mind cases in two-space and one-time dimensions.

A direct search for particular solutions is possible as shown in [17] where the so-called tanh method is utilized for a special cubic model. However, with this technique, calculations become rapidly cumbersome and are not suitable for model classification. Just to give an example, we considered the cubic model

$$u_t = \alpha u_{xx} - u^2 w + z_1 u + z_2 w + z_3, \quad (8.1a)$$

$$w_t = \beta w_{xx} - u^2 w + h_1 u + h_2 w + h_3, \quad (8.1b)$$

and looked for a travelling wave with speed v . In the case $\alpha = \beta = v^2 z_1^2 / (2z_3^2)$, $h_1 = z_1 + z_2$, $h_2 = 0$, $h_3 = z_3 / z_1 (z_1 + z_2)$, we were able to find the solution

$$u = \sqrt{z_1 + z_2} \tanh \left(\frac{\xi}{v} \frac{z_3}{z_1} \sqrt{z_1 + z_2} \right), \quad (8.2a)$$

$$w = \frac{z_3}{z_1} + u, \quad (8.2b)$$

which, in the special case $z_2 = 0$, reproduces one of the solutions in [17].

To conclude our comments, we notice that mathematical models are only a rough simplification of a complex reality where the mechanisms of the biological, chemical and physical processes involved are often unknown. Notwithstanding, we think that an important role of these models is to get insight into possible interactions between specific processes and to suggest sometime new experiments [18].

ACKNOWLEDGMENTS

APPENDIX A: SOLUTION OF A SECOND ORDER DIFFERENTIAL EQUATION

The solution of Eq. (3.36) can be achieved via a trick by Mittag-Leffler [19]. In doing so, if $\tilde{Y}(\xi)$ is a solution of the equation

$$\tilde{Y}'' = a\tilde{Y}' - \frac{6}{25}a^2\tilde{Y} + c\tilde{Y}^2, \quad (\text{A1})$$

with a, c arbitrary constants, then

$$H(\xi) = \left(\tilde{Y}' - \frac{2}{5}a\tilde{Y}\right)^2 - \frac{2}{3}c\tilde{Y}^3, \quad (\text{A2})$$

fulfills the equation

$$H' = \frac{6}{5}aH. \quad (\text{A3})$$

Thus, we can find a constant H_0 such that

$$\left(\tilde{Y}' - \frac{2}{5}a\tilde{Y}\right)^2 - \frac{2}{3}c\tilde{Y}^3 = H_0 \exp\left(\frac{6}{5}a\xi\right). \quad (\text{A4})$$

By making in (A4) the change of variables

$$\tilde{Y} = \exp\left(\frac{2}{5}a\xi\right) \varphi, \quad z = \frac{5}{a}\sqrt{\frac{c}{6}} \exp\left(\frac{a\xi}{6}\right), \quad (\text{A5})$$

we get

$$\varphi'^2(z) = 4\varphi^3(z) - k, \quad k \text{ constant}, \quad (\text{A6})$$

wich is solved by

$$\varphi(z) = \mathcal{P}(z - z_0, 0, k). \quad (\text{A7})$$

APPENDIX B: PROLONGATION OF THE QUADRATIC MODEL

In theory, quadratic reaction functions are compatible with linearizability. However, algebraic constraints have to be satisfied. As we shall see, the quadratic Gierer-Meinhardt model does not fall into the linearizable class, but showing this is not trivial.

a. *The non-degenerate case: $\alpha + \beta \neq 0$*

Repeating the same computation done in the case of the non-degenerate cubic model we find the following structure for the prolongation problem

$$y_x = F = \frac{1}{\alpha}a_1u + \frac{1}{\beta}a_2w + a_3, \quad (\text{B1a})$$

$$y_t = G = a_1u_x + a_2w_x - [a_1, a_3]u - [a_2, a_3]w + a_4, \quad (\text{B1b})$$

The reaction equation

$$F_u R_1 + F_w R_2 + [F, -[a_1, a_3]u - [a_2, a_3]w + a_4] = 0, \quad (\text{B2a})$$

$$R_1 = \epsilon(uw - u), \quad R_2 = \lambda(1 - uw), \quad (\text{B2b})$$

leads to the following algebra

$$[a_1, a_2] = 0, \quad (\text{B3a})$$

$$[a_1, [a_1, a_3]] = 0, \quad (\text{B3b})$$

$$[a_2, [a_2, a_3]] = 0, \quad (\text{B3c})$$

$$[a_1, a_4] = \epsilon a_1 + \alpha [a_3, [a_1, a_3]], \quad (\text{B3d})$$

$$[a_2, a_4] = \beta [a_3, [a_2, a_3]], \quad (\text{B3e})$$

$$[a_3, a_4] = -\frac{\lambda}{\beta}a_2, \quad (\text{B3f})$$

$$\frac{1}{\alpha} [a_1, [a_2, a_3]] + \frac{1}{\beta} [a_2, [a_1, a_3]] = \frac{\epsilon}{\alpha}a_1 - \frac{\lambda}{\beta}a_2. \quad (\text{B3g})$$

Let us scale the generators

$$a_1/\alpha \rightarrow a_1, \quad a_2/\beta \rightarrow a_2, \quad (\text{B4})$$

in order to cast the incomplete algebra into the simpler form

$$[a_1, a_2] = 0, \quad (\text{B5a})$$

$$[a_1, [a_1, a_3]] = 0, \quad (\text{B5b})$$

$$[a_2, [a_2, a_3]] = 0, \quad (\text{B5c})$$

$$[a_1, a_4] = \epsilon a_1 + \alpha [a_3, [a_1, a_3]], \quad (\text{B5d})$$

$$[a_2, a_4] = \beta [a_3, [a_2, a_3]], \quad (\text{B5e})$$

$$[a_3, a_4] = -\lambda a_2, \quad (\text{B5f})$$

$$[a_1, [a_2, a_3]] + [a_2, [a_1, a_3]] = \epsilon a_1 - \lambda a_2. \quad (\text{B5g})$$

Since $[a_1, a_2] = 0$ we can write

$$[a_1, [a_2, a_k]] = [a_2, [a_1, a_k]], \quad \forall k, \quad (\text{B6})$$

and Eq. (B5g) may be rewritten as

$$[a_1, [a_2, a_3]] = [a_2, [a_1, a_3]] = \frac{1}{2}(\epsilon a_1 - \lambda a_2). \quad (\text{B7})$$

Moreover, from Eq. (B5d) and Eq. (B5e) we find

$$[a_2, [a_1, a_4]] = \alpha [a_2, [a_3, [a_1, a_3]]], \quad (\text{B8a})$$

$$[a_1, [a_2, a_4]] = \beta [a_1, [a_3, [a_2, a_3]]]. \quad (\text{B8b})$$

Then, using again Eq. (B6) we get

$$\alpha [a_2, [a_3, [a_1, a_3]]] = \beta [a_1, [a_3, [a_2, a_3]]], \quad (\text{B9})$$

from which

$$\alpha [a_3, [a_2, [a_1, a_3]]] - \alpha [[a_1, a_3], [a_2, a_3]] = \quad (\text{B10a})$$

$$= \beta [a_3, [a_1, [a_2, a_3]]] - \beta [[a_2, a_3], [a_1, a_3]], \quad (\text{B10b})$$

or

$$[[a_1, a_3], [a_2, a_3]] = \frac{1}{2} \frac{\beta - \alpha}{\beta + \alpha} (\epsilon [a_1, a_3] - \lambda [a_2, a_3]), \quad (\text{B11})$$

by virtue of the Jacobi identity. Now if we substitute this result into the Jacobi identity among the three operators

$$a_1, \quad [a_1, a_3], \quad [a_2, a_3], \quad (\text{B12})$$

we obtain $[a_1, [a_2, a_3]] = 0$ and consequently

$$\epsilon a_1 = \lambda a_2. \quad (\text{B13})$$

This linear dependence between a_1 and a_2 constrains the incomplete algebra to close collapsing onto $\text{sim}(2)$. In particular, complete linearizability is not achieved.

b. The degenerate case: $\alpha + \beta = 0$

In the degenerate case, after proper rescaling, the incomplete algebra turns out to be

$$[a_1, a_4] = [a_3, [a_1, a_3]] + \epsilon a_1 - \lambda a_0, \quad (\text{B14a})$$

$$[a_1, [a_1, a_3]] = 0 = [a_2, [a_2, a_3]], \quad (\text{B14b})$$

$$[a_2, a_4] = [a_3, [a_3, a_2]], \quad (\text{B14c})$$

$$[a_0, a_4] = -2 [a_3, [a_1, a_2]] + \epsilon a_0 - \epsilon a_1 - \lambda a_2, \quad (\text{B14d})$$

$$[a_1, [a_1, a_2]] = \lambda a_0, \quad (\text{B14e})$$

$$[a_2, [a_1, a_2]] = \epsilon a_0, \quad (\text{B14f})$$

$$[a_3, a_4] = \lambda a_2, \quad (\text{B14g})$$

$$[a_0, a_i] = 0, \quad i = 1, 2, 3. \quad (\text{B14h})$$

It is easy to show that a_0 must vanish. In order to see this we write

$$[a_1, [a_0, a_4]] = -2[a_1, [a_3, [a_1, a_2]]] - \lambda[a_1, a_2] = 2[[a_1, a_2], [a_1, a_3]] - \lambda[a_1, a_2], \quad (\text{B15})$$

but the left hand side vanishes because

$$[a_1, [a_0, a_4]] = [a_0, [a_1, a_4]] - [a_4, [a_1, a_0]] = [a_0, [a_3, [a_1, a_3]] + \epsilon a_1 - \lambda a_0] = 0. \quad (\text{B16})$$

Hence

$$[[a_1, a_2], [a_1, a_3]] = \frac{\lambda}{2}[a_1, a_2], \quad (\text{B17})$$

and

$$\begin{aligned} \lambda a_0 &= [a_1, [a_1, a_2]] = \frac{2}{\lambda}[a_1, [[a_1, a_2], [a_1, a_3]]] = \\ &= \frac{2}{\lambda}\{[[a_1, a_2], [a_1, [a_1, a_3]]] - [[a_1, a_3], [a_1, [a_1, a_2]]]\} = \\ &= -\frac{2}{\lambda}[[a_1, a_3], a_0\lambda] = 0. \end{aligned} \quad (\text{B18a})$$

The algebra simplifies and becomes

$$[a_1, a_4] = [a_3, [a_1, a_3]] + \epsilon a_1, \quad (\text{B19a})$$

$$[a_1, [a_1, a_3]] = 0 = [a_2, [a_2, a_3]], \quad (\text{B19b})$$

$$[a_2, a_4] = [a_3, [a_3, a_2]], \quad (\text{B19c})$$

$$[a_1, [a_1, a_2]] = 0, \quad (\text{B19d})$$

$$[a_2, [a_1, a_2]] = 0, \quad (\text{B19e})$$

$$[a_3, a_4] = \lambda a_2, \quad (\text{B19f})$$

$$[a_3, [a_1, a_2]] = -\frac{1}{2}(\epsilon a_1 + \lambda a_2). \quad (\text{B19g})$$

Let us show that if the dimension of the space over which the above operator are defined is finite, then

$$[a_1, [a_1, a_2]] = 0 = [a_2, [a_1, a_2]] \Rightarrow [a_1, a_2] = 0, \quad (\text{B20})$$

from which, by using Eq. (B19g), we obtain the desired constraint

$$\epsilon a_1 + \lambda a_2 = 0. \quad (\text{B21})$$

The proof of Eq. (B20) goes as follows: let A, B, X be a triple of linear operators acting on a finite dimensional vector space, satisfying

$$[A, B] = X, \quad [A, X] = 0, \quad [B, X] = 0. \quad (\text{B22})$$

Obviously $\text{Tr} X = 0$. Furthermore, since

$$\text{Tr}(A[B, C]) = \text{Tr}(B[C, A]), \quad (\text{B23})$$

we obtain, for $n > 1$,

$$\text{Tr} X^n = \text{Tr}(X^{n-1} [A, B]) = \text{Tr}(A [B, X^{n-1}]) = 0. \quad (\text{B24})$$

Thus, the characteristic polynomial of X is

$$\det(X - \lambda) = (-\lambda)^{\dim X}, \quad (\text{B25})$$

and X has only the null eigenvalue. Let the canonical form of X be

$$\begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_N \end{pmatrix}, \quad (\text{B26})$$

(where Λ_n are n -dimensional Jordan matrices). The general form of the matrices which commute with (B26) is block-rectangular where every block is a Toeplitz upper triangular matrix [20]. Then, it is easy to show that the condition $[A, B] = X$ implies that all the Λ_n blocks are 1-dimensional and therefore $X = 0$.

APPENDIX C: THE SIMILITUDE GROUP AND CASIMIR OPERATORS

Let us recall that given a Lie algebra \mathcal{L} with generators T_α

$$[T_\alpha, T_\beta] = c_{\alpha\beta}^\gamma T_\gamma, \quad (\text{C1})$$

a Casimir operator \mathcal{C} is a polynomial in $\{T_\alpha\}$ which commutes with all the generators

$$[\mathcal{C}, T_\alpha] = 0. \quad (\text{C2})$$

If the Lie algebra is semisimple, the metric tensor

$$g_{\alpha\beta} = c_{\alpha\lambda}^\tau c_{\beta\tau}^\lambda, \quad (\text{C3})$$

is non singular and the possible Casimir operators are contained in the sequence

$$I_2 = c_{\alpha_1\beta_1}^{\beta_2} c_{\alpha_2\beta_2}^{\beta_1} T^{\alpha_1} T^{\alpha_2}, \quad (\text{C4a})$$

$$I_3 = c_{\alpha_1\beta_1}^{\beta_2} c_{\alpha_2\beta_2}^{\beta_3} c_{\alpha_3\beta_3}^{\beta_1} T^{\alpha_1} T^{\alpha_2} T^{\alpha_3}, \quad (\text{C4b})$$

$$\dots \quad (\text{C4c})$$

where

$$T^\alpha = g^{\alpha\beta} T_\beta, \quad (\text{C5a})$$

$$g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha. \quad (\text{C5b})$$

In the non semisimple case (for instance $\text{sim}(2)$), the independent Casimir operators must be built explicitly. Their number N is expressed by

$$N = \dim \mathcal{L} - \max_{a_1, \dots, a_{\dim \mathcal{L}}} \text{rank} \|c_{\sigma\tau}^\rho a_\rho\|, \quad (\text{C6})$$

according to the Beltrametti-Blasi theorem [21]. In the particular case of $sim(2)$ we find $N = 0$ and no Casimir operator exists. On the other hand, the Euclidean subalgebra generated by A_1 , A_3 and A_5 admits a Casimir operator

$$\mathcal{C} = A_1^2 + A_5^2, \quad (C7)$$

and since

$$[\mathcal{C}, A_4] = 2\mathcal{C}, \quad (C8)$$

it is easy to show that given a solution y to the linearized problem

$$y_x = F_1 A_1 y + F_3 A_3 y, \quad (C9a)$$

$$y_t = G_1 A_1 y + G_5 A_5 y + \nu A_4 y, \quad (C9b)$$

then the new pseudopotential

$$\tilde{y} = e^{-2\nu t} \mathcal{C} y, \quad (C10)$$

is another solution corresponding to the same (u, w) appearing in the functions F and G .

APPENDIX D: HEURISTIC DETERMINATION OF α^*

Let us look for a solution to the equation

$$\theta' = -\theta + \xi \theta^2 + \xi \theta^3, \quad (D1)$$

with

$$\theta(0) = \alpha > 0, \quad (D2)$$

in the following form

$$\theta(\xi) = \sum_{n=1}^{\infty} e^{-n\xi} P_n(\xi), \quad P_1(\xi) = \alpha, \quad P_{n<1}(\xi) \equiv 0, \quad (D3)$$

where P_n are functions to be determined. Choosing the integration constants in a sensible way, we obtain

$$P_n(\xi) = e^{(n-1)\xi} \int_{+\infty}^{\xi} e^{-(n-1)t} t \left\{ \sum_{m=1}^{n-1} P_m(t) P_{n-m}(t) + \sum_{m=1, l=1}^{n-1} P_m P_l P_{n-m-l} \right\}. \quad (D4)$$

The functions P_n are polynomials, the first two being $P_1 = \alpha$, $P_2 = -\alpha^2(\xi + 1)$. The initial value $\theta(0)$ is an unknown function of α admitting the series expansion

$$\theta(0) = \sum_{n>1} P_n(0) = \alpha - \alpha^2 + \frac{3}{4}\alpha^3 - \frac{17}{54}\alpha^4 - \frac{605}{3456}\alpha^5 + \dots = \sum b_n \alpha^n. \quad (D5)$$

This series shows a poor convergence near $\alpha \sim 1$, but in terms of the mapped variable

$$z = \frac{\alpha}{\alpha + 1}, \quad (\text{D6})$$

we find

$$\theta(0) = z - \frac{1}{4}z^3 - \frac{7}{108}z^4 + \dots = \sum b'_n z^n. \quad (\text{D7})$$

If we assume this series to have an infinite convergence radius and plot the function

$$\sum b'_n \left(\frac{\alpha}{\alpha + 1} \right)^n, \quad (\text{D8})$$

we see that $\theta(0)$ appears to be upper bounded by a finite constant α^* whose numerical value can be estimated by truncating the infinite coefficients b'_n . Taking the first 20 coefficients, we find

$$\alpha^* = 0.756561. \quad (\text{D9})$$

APPENDIX E: ANALYSIS OF THE ROOTS OF A QUARTIC EQUATION

If $Q(x)$ is a polynomial of degree N , then the number Z of roots on the left side of the imaginary axis is given by

$$Z = \frac{N}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re} \left. \frac{Q'(z)}{Q(z)} \right|_{z=iy} dy. \quad (\text{E1})$$

In the case $Q(x) = 2x^4 - 3vx^3 + (v^2 - 1)x^2 + 1$ we have

$$Z = 2 + \frac{1}{2\pi} \int_{-\infty}^{\infty} H(y) dy, \quad (\text{E2})$$

where

$$H(y) = \frac{d}{dy} \left\{ \text{Arctan} \frac{3vy^3}{2y^4 + (1 - v^2)y^2 + 1} \right\}. \quad (\text{E3})$$

Since $\text{Arctan}(\dots) \rightarrow 0$ as $|y| \rightarrow 0$ we can conclude that $Z = 2$. The Arctan argument is odd and its possible singularities give contributions which cancel in pairs.

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FIGURES

FIG. 1. Exact solutions for the u, w fields in the four cases corresponding to Eqs. (3.23), (3.30), (3.38) and (3.40). The numerical values of the parameters are respectively i) $v = 1, \alpha_1 = 1, \alpha_2 = 0, \lambda_1 = 1, \lambda_2 = -1, \mu_1 = 1, \zeta = 1$; ii) $v = 1, \alpha_1 = 1, \alpha_2 = -1, \lambda_1 = 1, \lambda_2 = 1, \mu_1 = 1, \zeta = 1$; iii) $v = 1, \alpha_1 = 2, \alpha_2 = 1, \lambda_1 = 1, \lambda_2 = -2, \mu_1 = 1, \zeta = 0$; iv) $v = 1, \alpha_1 = 1, \alpha_2 = -1, \lambda_1 = 1, \lambda_2 = 1, \mu_1 = 1, \zeta = 1, \rho_1 = 1$.

FIG. 2. Qualitative behaviour of Eq. (6.21).

FIG. 3. Analytical integrals of Eq. (6.31).

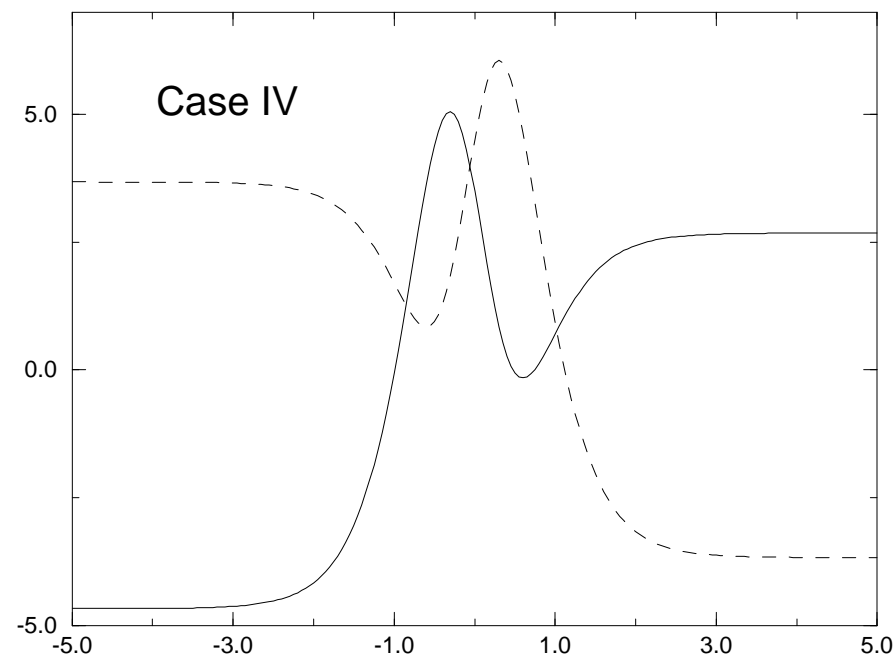
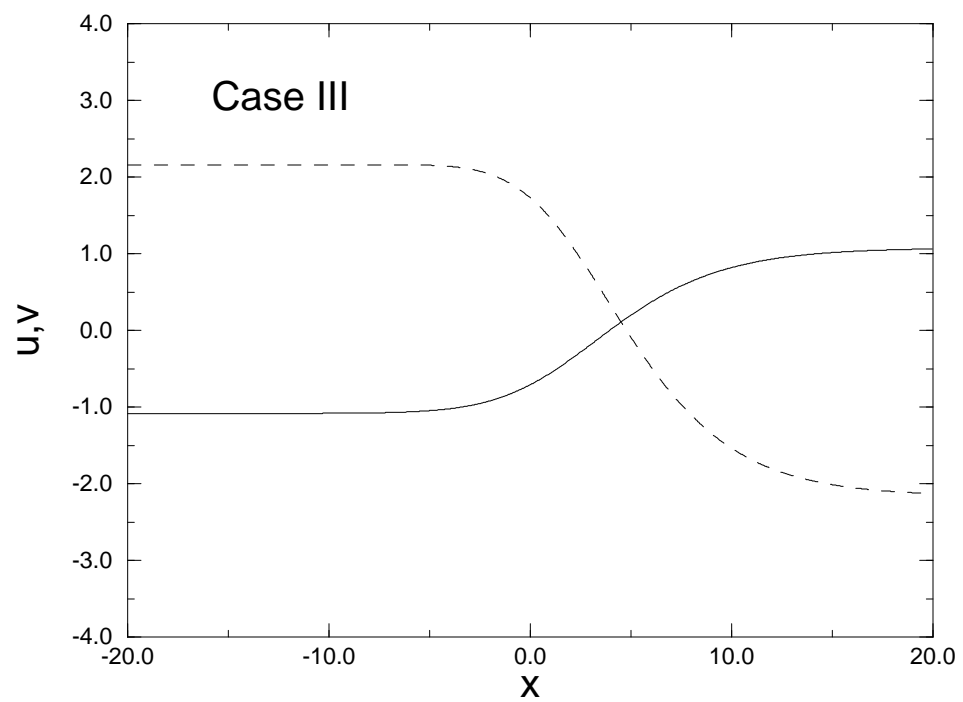
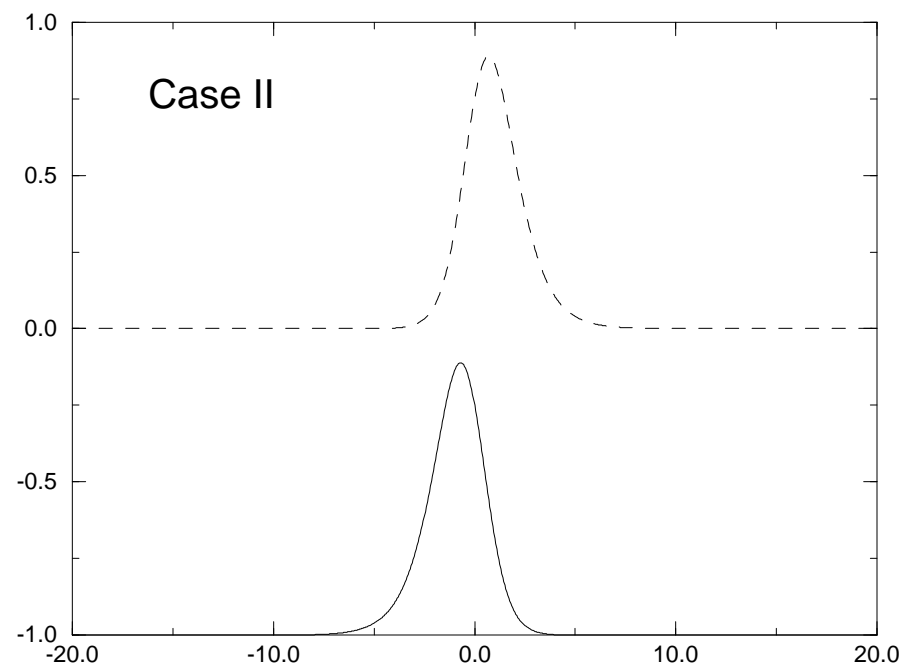
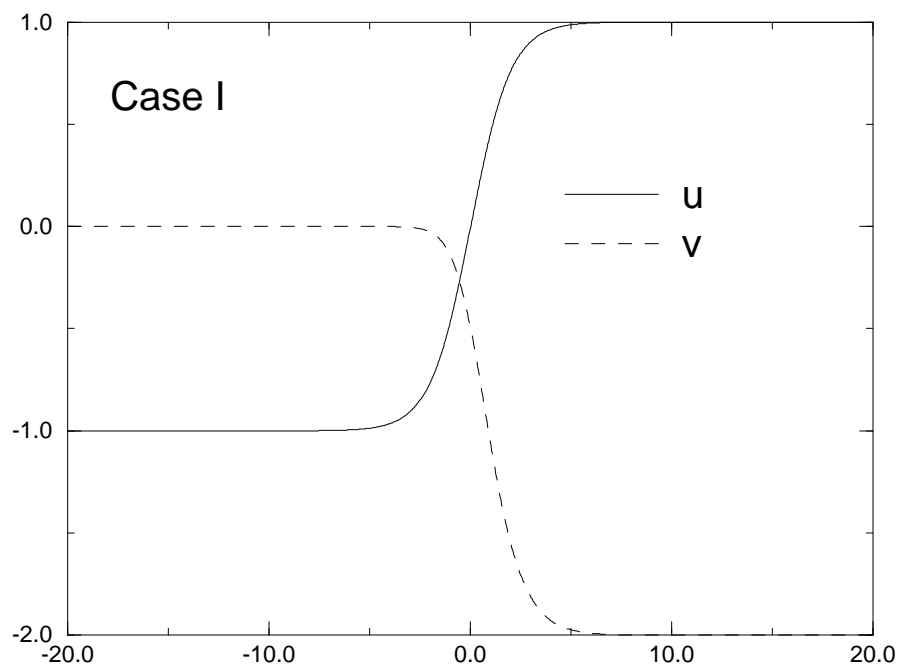
FIG. 4. Solution corresponding to Eqs (6.42) and (6.43).

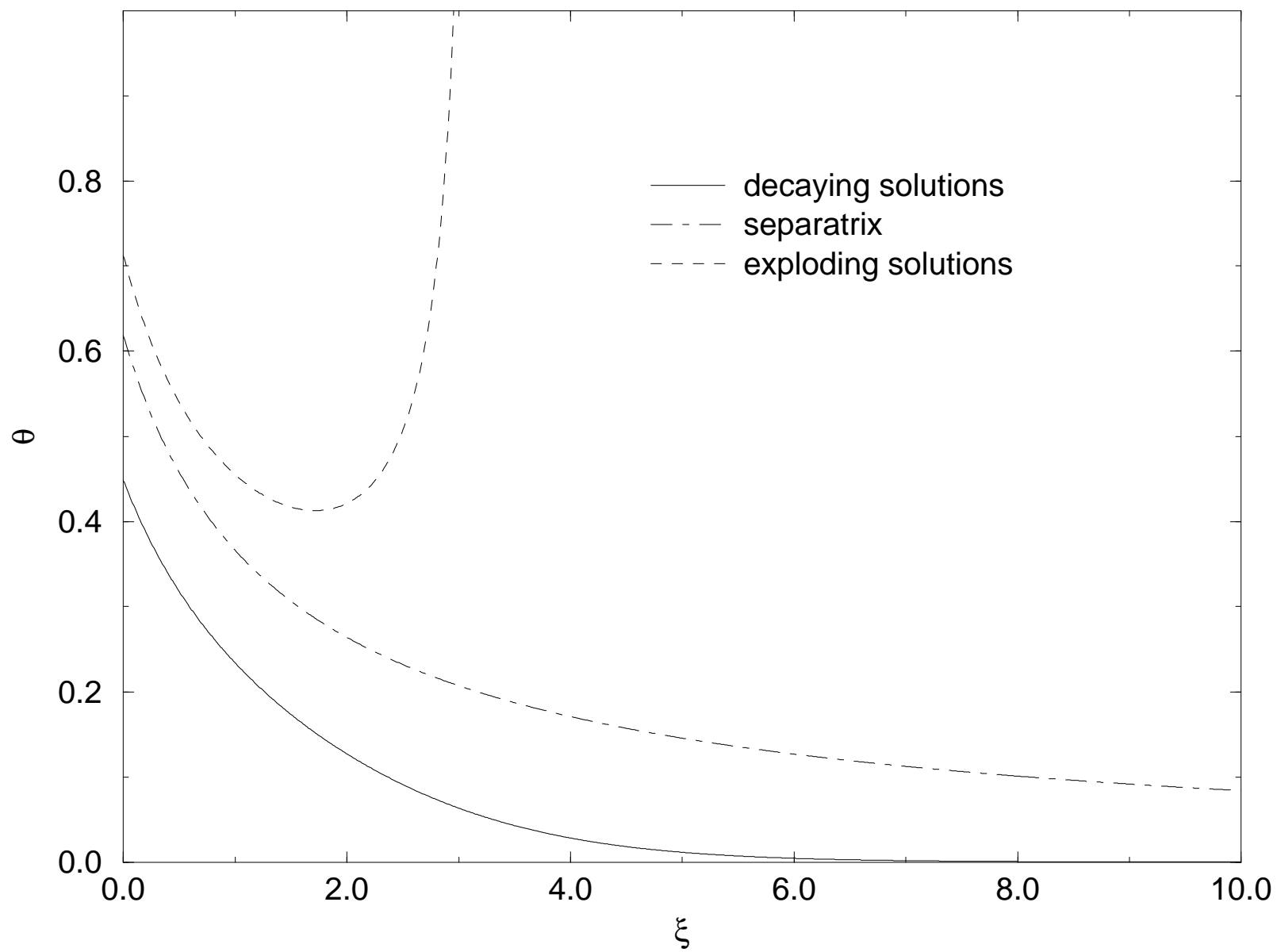
FIG. 5. Homogeneous solutions obtained from numerical integration of Eqs. (7.2). We remark that from the value of α^* given in Eq. (D9) we get $\alpha^*/(1 + \alpha^*) \simeq 0.431$.

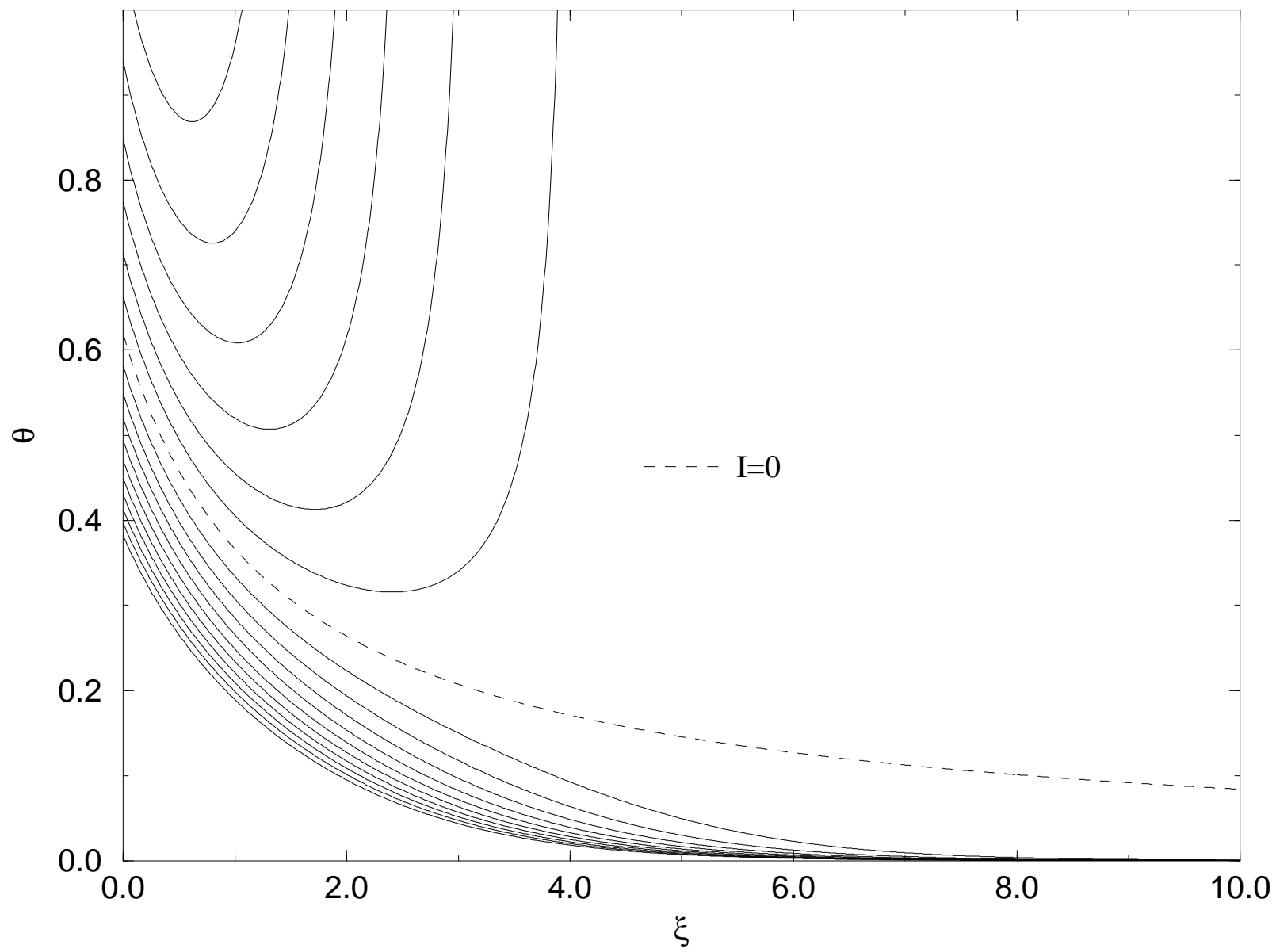
FIG. 6. Evolution of the approximate travelling wave with $v = 2.0$. The values of the other parameters are $\alpha = 2, \beta = 1, \epsilon = 1, \lambda = 1, \Delta x = 0.05, \Delta t = 0.00025$, Δx and Δt being the space and time discretization steps.

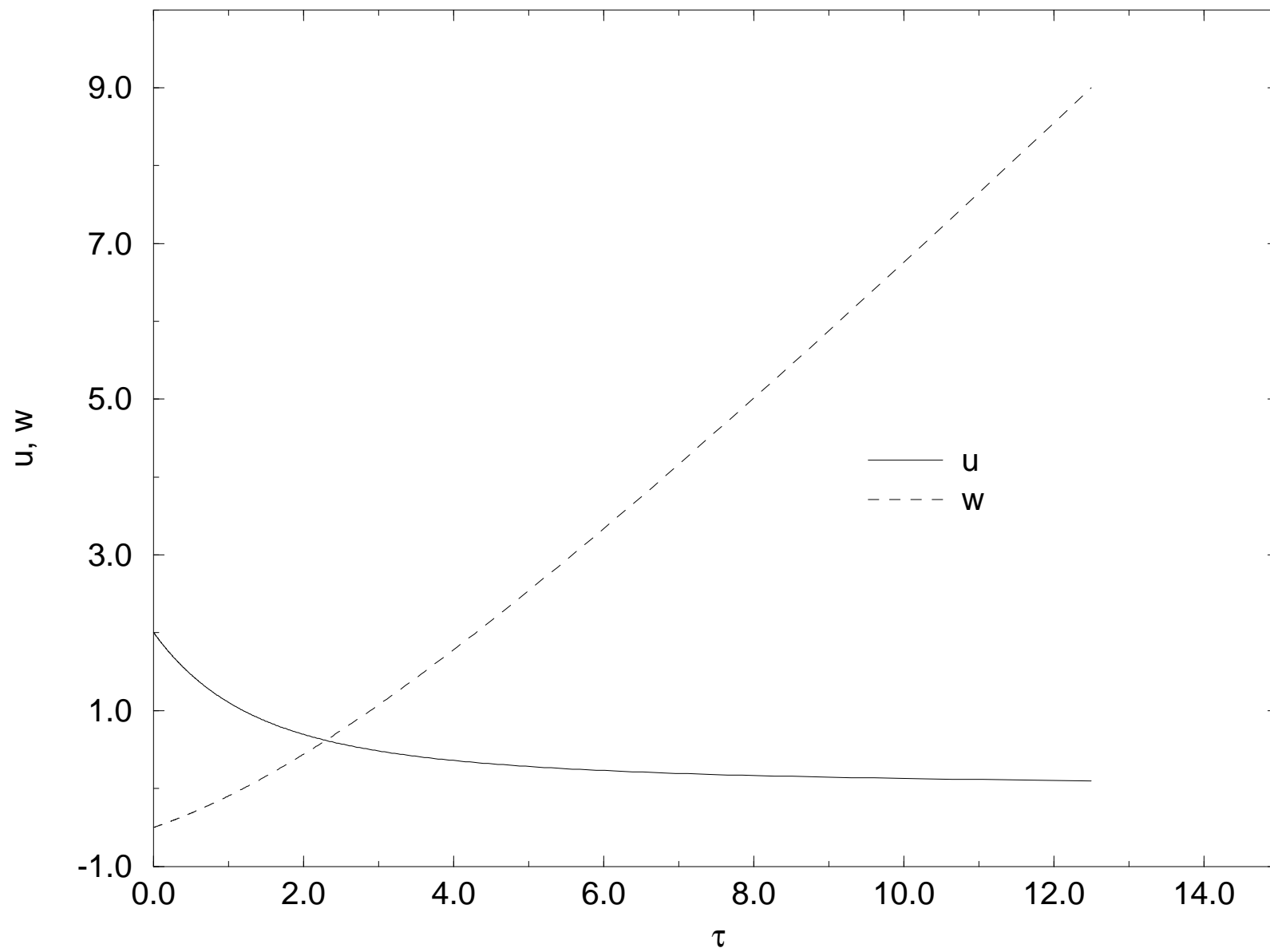
FIG. 7. Like Fig. 6, but $v = 1.0$.

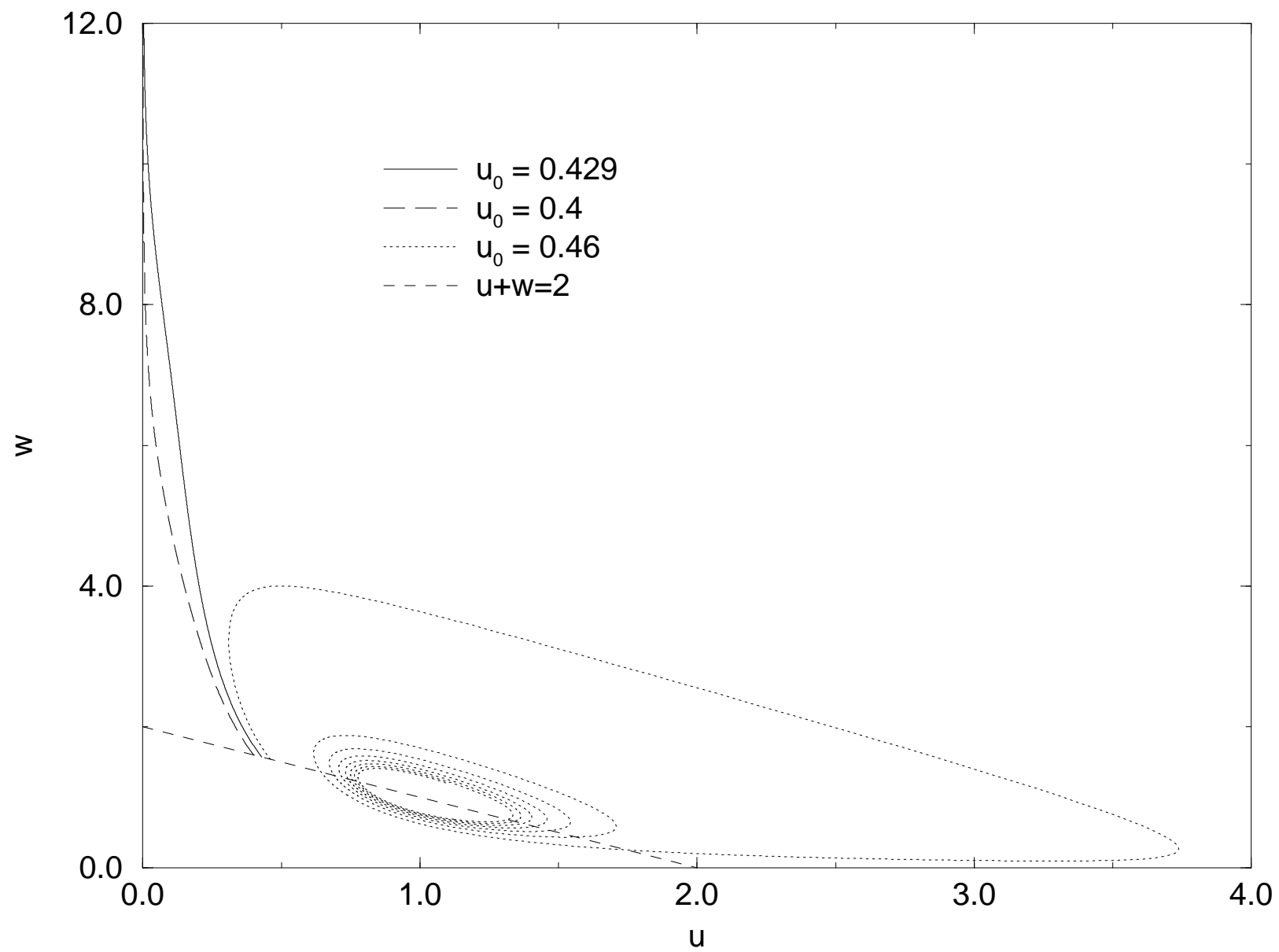
FIG. 8. Like Fig. 6, but $v = 0.5$.











$\nu = 2.0$

